

# Weak approximation and rationally connected varieties over function fields of curves

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## Introduction

This paper surveys recent work on weak approximation for varieties over complex function fields. It also touches on the geometric theory of rationally connected varieties and stable maps.

Weak approximation has been studied extensively in the context of number theory, quadratic forms, and linear algebraic groups. Early examples include work of Kneser [Kne62] and Harder [Har68] on linear algebraic groups over various fields. In the 1980's, attention shifted to rational surfaces over number fields and cohomological obstructions to weak approximation. Significant results were obtained by Colliot-Thélène, Sansuc, Swinnerton-Dyer, Skorobogatov, Salberger, Harari, and others. We refer the reader to [Har04] for an excellent survey of the state of this area in 2002.

With the development of the theory of rationally connected varieties, weak approximation over function fields of *complex* curves became a focus of research. Already in 1992, Kollár-Miyaoka-Mori [KMM92b] showed that rationally connected varieties over such fields enjoy remarkable approximation properties, assuming they admit rational points. In 2001, Graber-Harris-Starr [GHS03] showed these rational points exist, which opened the door to a more systematic study of their properties.

This paper is organized as follows: Section 1 reviews the basic definitions, presenting them in a form useful for our purposes. In Section 2, we

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present results valid for general rationally connected varieties, as well as key constructions and deformation-theoretic tools. We turn to special classes of varieties in Section 3, including rational surfaces and hypersurfaces with mild singularities at places of bad reduction. Section 4 addresses a large class of varieties where weak approximation is known, the rationally simply connected varieties [dJS06]. We raise some questions for further study in Section 5. The Appendix presents basic facts on stable maps used throughout the volume.

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## 1 Elements of weak approximation

**Notation** Throughout, a *variety* over a field  $L$  designates a separated geometrically integral scheme of finite type over  $L$ ; its *generic point* is the unique point corresponding to its function field. A *general point* of a variety is a closed point chosen from the complement of an unspecified Zariski closed proper subset.

Let  $k$  be an algebraically closed field of characteristic zero and  $B$  a smooth projective curve over  $k$ , with function field  $F = k(B)$ .

Let  $X$  be a smooth projective variety over  $F$ . A *model* of  $X$  is a flat proper morphism  $\pi : \mathcal{X} \rightarrow B$  with generic fiber  $X$ . Usually,  $\mathcal{X}$  is a scheme

projective over  $B$ , but there are situations where we should take it to be an algebraic space proper over  $B$ . For each  $b \in B$ , let  $\mathcal{X}_b = \pi^{-1}(b)$  denote the fiber over  $b$ . Once we have chosen a concrete embedding  $X \subset \mathbb{P}^N$ , the properness of the Hilbert scheme yields a natural model. The model is *regular* if the total space  $\mathcal{X}$  is nonsingular; this can always be achieved via resolution of singularities.

**Elementary properties of sections** Recall that a *section* of  $\pi$  is a morphism  $s : B \rightarrow \mathcal{X}$  such that  $\pi \circ s : B \rightarrow B$  is the identity. By the valuative criterion of properness, we have

$$\{\text{sections } s : B \rightarrow \mathcal{X} \text{ of } \pi\} \Leftrightarrow \{\text{rational points } x \in X(F)\}.$$

Assume  $\mathcal{X}$  is regular and write

$$\begin{aligned} \mathcal{X}^{sm} &= \{x \in \mathcal{X} : \pi \text{ is smooth at } x\} \\ &= \{x \in \mathcal{X} : \mathcal{X}_b \text{ is smooth at } x, b = \pi(x)\} \subset \mathcal{X}. \end{aligned} \tag{1}$$

Then each section  $s : B \rightarrow \mathcal{X}$  is necessarily contained in  $\mathcal{X}^{sm}$ . The proof of this assertion is basic calculus: Since  $\pi \circ s$  is the identity the derivative  $d(\pi \circ s)$  is as well, which means that  $d\pi$  is surjective and

$$\dim \ker(d\pi_{s(b)}) = \dim \mathcal{X}_b$$

for each  $b \in B$ . Thus  $\mathcal{X}_b$  is smooth at  $s(b)$ .

**Formulating weak approximation** For each  $b \in B$ , let  $\widehat{\mathcal{O}}_{B,b}$  denote the completion of the local ring  $\mathcal{O}_{B,b}$  at the maximal ideal  $\mathfrak{m}_{B,b}$ , and  $\widehat{F}_b$  the completion of  $F = k(B)$  at  $b$ , i.e., the quotient field of  $\widehat{\mathcal{O}}_{B,b}$ . Consider the *adèles* over  $F$

$$\mathbb{A}_F = \prod'_{b \in B} \widehat{F}_b,$$

i.e., the restricted product over all the places of  $B$ . The restricted product means that all but finitely many of the factors are in  $\widehat{\mathcal{O}}_{B,b}$ . There are *two* natural topologies one could consider: The ordinary product topology and the restricted product topology, with basis consisting of products of open sets  $\prod_{b \in B} U_b$ , where  $U_b = \widehat{\mathcal{O}}_{B,b}$  for all but finitely many  $b$ . Using the natural inclusions  $F \subset \widehat{F}_b$ , we may regard  $F \subset \mathbb{A}_F$ .

Given a variety  $X$  over  $F$ , the adelic points  $X(\mathbb{A}_F)$  inherit both topologies from  $\mathbb{A}_F$ .

**Definition 1.1.** A variety  $X$  over  $F$  satisfies *weak approximation* (resp. *strong approximation*) if

$$X(F) \subset X(\mathbb{A}_F)$$

is dense in the ordinary product (resp. restricted product) topology.

For *proper* varieties  $X$ , the distinction between weak and strong approximation is irrelevant. This will be clear after we analyze the definitions in this case.

**Unwinding the definition** Assume  $X$  is smooth and proper and fix a model  $\pi : \mathcal{X} \rightarrow B$ . Both topologies on  $X(\mathbb{A}_F)$  have the following basis: Consider data

$$J = (N; b_1, \dots, b_r; \hat{s}_1, \dots, \hat{s}_r),$$

consisting of a nonnegative integer  $N$ , distinct places  $b_1, \dots, b_r \in B$ , and points  $\hat{s}_i \in X(\widehat{F}_{b_i})$  for  $i = 1, \dots, r$ . Since  $\pi : \mathcal{X} \rightarrow B$  is proper, we may interpret  $\hat{s}_i$  as a section of the restriction

$$\pi|_{\widehat{B}_{b_i}} : \mathcal{X} \times_B \widehat{B}_{b_i} \rightarrow \widehat{B}_{b_i}, \quad \widehat{B}_{b_i} = \text{Spec}(\widehat{\mathcal{O}}_{B,b_i}),$$

to the completion of  $B$  at  $b_i$ . Since we can freely clear denominators, insisting that the points are integral at almost all places is not a restriction. Thus our basic open sets are

$$U_J = \{t \in X(\mathbb{A}_F) : t \equiv \hat{s}_i \pmod{\mathfrak{m}_{B,b_i}^{N+1}}\},$$

i.e., sections with Taylor series at  $b_1, \dots, b_r$  prescribed to order  $N$ .

Now suppose in addition that  $\pi : \mathcal{X} \rightarrow B$  is a *regular* model, so that sections automatically factor through  $\mathcal{X}^{sm} \subset \mathcal{X}$  (see (1) above). Note that  $\hat{s}_i(b_i) \in \mathcal{X}_{b_i}$  is a smooth point, by the same calculus argument we used to show sections factor through  $\mathcal{X}^{sm}$ . Conversely, Hensel's Lemma (or the  $\mathfrak{m}$ -adic version of Newton's method, cf. [Ser73, p.14]) implies that each section  $\hat{s}_i^N$  of

$$\mathcal{X}^{sm} \times_B \text{Spec}(\mathcal{O}_{B,b_i}/\mathfrak{m}_{B,b_i}^{N+1}) \rightarrow \text{Spec}(\mathcal{O}_{B,b_i}/\mathfrak{m}_{B,b_i}^{N+1})$$

can be extended to a section  $\hat{s}_i$  of  $\pi|_{\widehat{B}_{b_i}}$ . Thus we can recast our data as a collection of *jet data*

$$J = (N; b_1, \dots, b_r; \hat{s}_1^N, \dots, \hat{s}_r^N), \tag{2}$$

where the  $\hat{s}_i^N$  are  $N$ -jets of sections of  $\mathcal{X}^{sm} \rightarrow B$  at  $b_i$ .

To summarize:

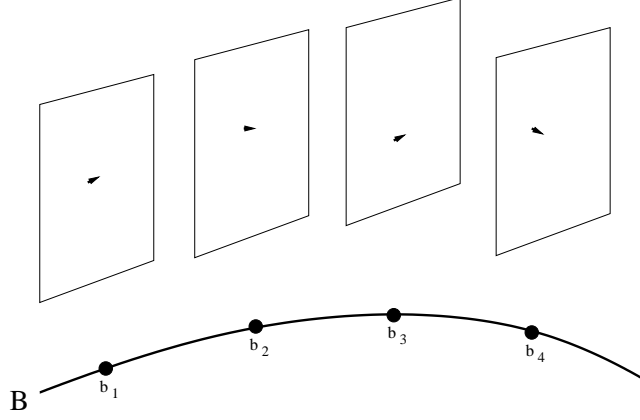


Figure 1: Jet data for sections

*Observation 1.2.* Let  $X$  be a smooth proper variety over  $F$ . To establish weak approximation for  $X$ , it suffices to show, for *one* regular model  $\mathcal{X} \rightarrow B$ , that for each collection of jet data (2) there exists a section  $s : B \rightarrow \mathcal{X}$  with  $s \equiv \hat{s}_i^N \pmod{\mathfrak{m}_{B, b_i}^{N+1}}$ .

**Iterated blow-ups arising from formal sections** Let  $X$  be a smooth proper variety over  $F$  of dimension  $d$ , with regular model  $\pi : \mathcal{X} \rightarrow B$ . Fix a point  $b \in B$  and a formal section

$$\hat{s} : \widehat{B}_b \rightarrow \mathcal{X} \times_B \widehat{B}_b.$$

This is equivalent to a point of  $X(\widehat{F}_b)$ .

We define a sequence of new models

$$\mathcal{X}^N \rightarrow \mathcal{X}^{N-1} \rightarrow \cdots \rightarrow \mathcal{X}^1 \rightarrow \mathcal{X}^0 = \mathcal{X}$$

inductively as follows:

1. Set  $\mathcal{X}^1 = \text{Bl}_{\hat{s}(b)}(\mathcal{X}^0)$  and let

$$\hat{t}^1 : \widehat{B}_b \rightarrow \mathcal{X}^1 \times_B \widehat{B}_b$$

denote the induced section, which exists by applying the valuative criterion to  $\mathcal{X}^1 \rightarrow B$ .

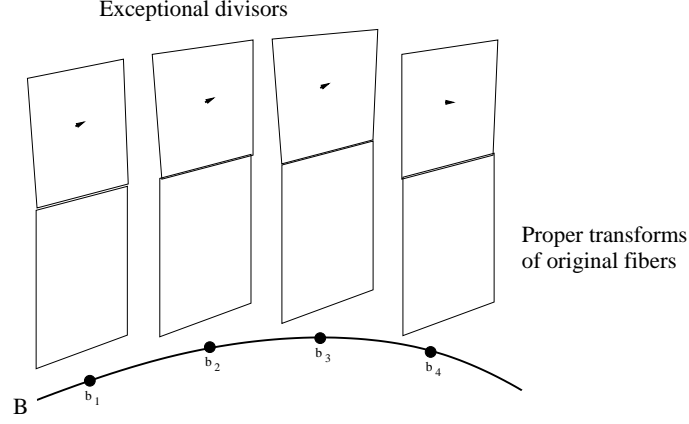


Figure 2: Iterated blow-up construction

2. Set  $\mathcal{X}^2 = \text{Bl}_{\hat{t}^1(b)}(\mathcal{X}^1)$  and

$$\hat{t}^2 : \widehat{B}_b \rightarrow \mathcal{X}^2 \times_B \widehat{B}_b$$

the induced section.

$\vdots$

N. Set  $\mathcal{X}^N = \text{Bl}_{\hat{t}^{N-1}(b)}(\mathcal{X}^{N-1})$  and

$$\hat{t}^N : \widehat{B}_b \rightarrow \mathcal{X}^N \times_B \widehat{B}_b$$

the induced section.

In other words, we blow up successively along the proper transforms of the formal section over  $b$ . Note that  $\mathcal{X}^1$  depends only on  $\hat{s}(b) = \hat{s}^0 = \hat{s} \pmod{\mathfrak{m}_{B,b}}$ , and in general,  $\mathcal{X}^N$  depends only on the jet datum  $\hat{s}^{N-1} = \hat{s} \pmod{\mathfrak{m}_{B,b}^N}$ . The fiber over  $b$  can be expressed

$$\mathcal{X}_b^N = \text{Bl}_{\hat{s}(b)}(\mathcal{X}_b) \cup \text{Bl}_{\hat{t}^1(b)}(\mathbb{P}^d) \cup \cdots \cup \text{Bl}_{\hat{t}^{N-1}(b)}(\mathbb{P}^d) \cup \mathbb{P}^d, \quad (3)$$

i.e., as a chain with the proper transform of  $\mathcal{X}_b$  at one end, the  $N$ th exceptional divisor at the other end, and blow-ups of the intermediate exceptional divisors in between. Observe that  $\hat{t}^N(b) \in \mathbb{P}^d$ , the  $N$ th exceptional divisor.

**Definition 1.3.** Let  $X$  be a smooth proper variety over  $F$  with regular model  $\pi : \mathcal{X} \rightarrow B$ , and

$$J = (N; b_1, \dots, b_r; \hat{s}_1^N, \dots, \hat{s}_r^N)$$

a collection of jet data as in (2). The *iterated blow-up* associated to  $J$

$$\beta^J : \mathcal{X}^J \rightarrow \mathcal{X}$$

is obtained by blowing up  $N$  times along over each point  $b_j$ . For each  $i = 1, \dots, r$ , let  $x_i^J \in \mathcal{X}_{b_i}^J$  denote the evaluation of the formal section defining the blow-up over  $b_i$  at the closed point.

**Proposition 1.4.** *Retaining the notation of Definition 1.3, we have a natural bijection*

$$\left\{ \begin{array}{c} \text{sections } s : B \rightarrow \mathcal{X} \\ \text{with jet data } J \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} \text{sections } s^J : B \rightarrow \mathcal{X}^J \\ \text{with } s^J(b_i) = x_i^J \end{array} \right\}.$$

The direction  $\Rightarrow$  is induced by taking the proper transform of  $s$  in  $\mathcal{X}_J$ ; the direction  $\Leftarrow$  is given by setting  $s = \beta^J \circ s^J$ .

In other words, we can interpret weak approximation on  $\mathcal{X}$  in terms of finding sections with prescribed values over the various iterated blow-ups of  $\mathcal{X}$ . To summarize:

*Observation 1.5.* Let  $X$  be a smooth proper variety over  $F$ . To establish weak approximation for  $X$ , it suffices to show, for *each* regular model  $\mathcal{X} \rightarrow B$ , distinct places  $b_1, \dots, b_r \in B$ , and smooth points  $x_i \in \mathcal{X}_{b_i}^{sm}$ ,  $i = 1, \dots, r$ , there exists a section  $s : B \rightarrow \mathcal{X}$  with  $s(b_i) = x_i$  for each  $i$ .

**Weak approximation and birational models** Weak approximation is a birational property (see [Kne62, §2.1]):

**Theorem 1.6.** *Let  $X_1$  and  $X_2$  be smooth varieties over  $F$ . Assume they are birational over  $F$ . Then  $X_1$  satisfies weak approximation if and only if  $X_2$  satisfies weak approximation.*

**Corollary 1.7.** *Let  $X$  be a smooth variety, rational over  $F$ . Then  $X$  satisfies weak approximation.*

*Proof.* Since  $X_1$  and  $X_2$  are birational, they share a common Zariski open dense subset

$$X_1 \supset U \subset X_2.$$

We claim that  $U(\mathbb{A}_F)$  is dense in  $X_1(\mathbb{A}_F)$  and  $X_2(\mathbb{A}_F)$  in the ordinary product topology, or equivalently,

$$X_i(\mathbb{A}_F) \setminus U(\mathbb{A}_F), \quad i = 1, 2,$$

has trivial interior. Given a point  $x \in X_1(\mathbb{A}_F) \setminus U(\mathbb{A}_F)$ , we claim there exists a point  $u \in U(\mathbb{A}_F)$  that approximates  $x$  (in the direct product topology) arbitrarily closely. Precisely, consider distinct places  $b_1, \dots, b_r \in B$  and the corresponding projections

$$x_{b_j} \in X(\widehat{F}_{b_j}).$$

It suffices to show that for each  $N \geq 0$ , there exist

$$u_{b_j} \in U(\widehat{F}_{b_j})$$

such that  $u_{b_j} \equiv x_{b_j} \pmod{\mathfrak{m}_{B,b_j}^{N+1}}$ .

We take  $u_{b_j}$  to be an  $N$ th order approximation of  $x_{b_j}$ , chosen in such a way that it does *not* lie on  $X_1 \setminus U$  (regarded as a subscheme with the reduced induced scheme structure). Standard  $\mathfrak{m}$ -adic approximation techniques (cf. [Ser73, p. 14]) allow us to exhibit such points.  $\square$

*Remark 1.8.* In geometric terms, nonempty open subsets of  $X_1(\widehat{F}_{b_j})$  are Zariski dense in  $X$ . We can produce points intersecting  $X_1 \setminus U$  over  $b_j$  with arbitrarily high order.

## Places of good reduction and construction of models

**Definition 1.9.** Let  $X$  be a smooth proper variety over  $F$  with regular model  $\mathcal{X} \rightarrow B$ . The model has *good reduction* at  $b \in B$  if  $\mathcal{X}_b$  is smooth; otherwise, it has *bad reduction* at  $b$ . A place  $b \in B$  is of *good reduction* if there exists a regular model  $\pi : \mathcal{X} \rightarrow B$  with good reduction at  $b$ , and of *bad reduction* otherwise.

It is clear that each model has only finitely many places of bad reduction. A gluing argument gives:

**Proposition 1.10.** [HT06, Prop. 7] *Let  $X$  be a smooth proper variety over  $F$ . Then there exists an algebraic space  $\mathcal{X} \rightarrow B$  that is a regular model for  $X$  and is smooth over each place of good reduction.*

Such models are called *good models*.



**An example: pencils of cubic surfaces** We really do need to consider algebraic spaces to get good models:

Consider the  $\mathbb{P}^{19}$  parametrizing all cubic surfaces in  $\mathbb{P}^3$ , with discriminant hypersurface  $D \subset \mathbb{P}^{19}$ . The fundamental group of  $U = \mathbb{P}^{19} - D$  acts on the cohomology lattice via the full monodromy representation. The monodromy group is the Weyl group  $W(E_6)$ , acting on the primitive cohomology lattice of the cubic surface via the standard representation generated by reflections in the simple roots [Har79]. (These simple roots are differences of disjoint lines on the cubic surface.) There is a normal simple subgroup  $H \subset W(E_6)$  of index two and order 25920, corresponding to the elements acting on the lattice with determinant one.

Take a general line in  $\mathbb{P}^{19}$ , i.e., one intersecting  $D$  transversally. This family may be represented by the bihomogeneous equation

$$\mathcal{X} = \{sF(w, x, y, z) + tG(w, x, y, z) = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^3, \quad \deg(F) = \deg(G) = 3.$$

The resulting cubic surface fibration  $\pi : \mathcal{X} \rightarrow \mathbb{P}^1$  has the following properties:

1.  $\mathcal{X}$  is nonsingular and each singular fiber  $\mathcal{X}_b$  has a single ordinary double point, i.e., an isolated singularity with smooth projective tangent cone; étale locally we have

$$\mathcal{X}_b \sim \{x_0x_1 + x_2^2 = t = 0\} \subset \{x_0x_1 + x_2^2 = t\} \sim \mathcal{X}.$$

2. the local monodromy near each singular fiber is a reflection in a simple root, i.e., the vanishing cycle for the ordinary double point;
3.  $\pi$  has 32 singular fibers;
4. the monodromy representation is the full group  $W(E_6)$ .

The first assertion is just a restatement of the generality assumption; the second is the Picard-Lefschetz monodromy formula. The third property follows from a straightforward computation with Euler characteristics:  $\mathcal{X}$  is the blow-up of  $\mathbb{P}^3$  along the complete intersection  $F = G = 0$ . The last property follows from the Lefschetz hyperplane theorem for fundamental groups, applied to the open variety  $U$  [GM88, p. 150].

Let  $\{p_1, \dots, p_{32}\}$  be the intersection of  $D$  with our general line, i.e., the discriminant of  $\pi$ . Let  $B$  denote the double cover of  $\mathbb{P}^1$  at these points and

$$\pi' : \mathcal{X}' := \mathcal{X} \times_{\mathbb{P}^1} B \rightarrow B$$

the pullback to  $B$ . The local monodromy near the singular fibers of  $\pi'$  is now trivial; at the singular point of each such fiber,  $\mathcal{X}'$  has an ordinary threefold double point, étale locally isomorphic to  $\{x_0x_1 + x_2^2 = u^2\}$ .

The monodromy representation is an index two subgroup of  $W(E_6)$ , which does not contain the Picard-Lefschetz reflections associated to the degenerate fibers; this must be the simple group  $H$ . The restriction of the standard reflection representation of  $W(E_6)$  to  $H$  is still irreducible. There are no primitive classes fixed under  $H$  and  $\text{Pic}(\mathcal{X}'/B)$  is still generated by the anti-canonical class.

The family  $\pi'$  admits a simultaneous resolution

$$\begin{array}{ccc} \mathcal{Y} & \rightarrow & \mathcal{X}' \\ & & \downarrow \\ & & B, \end{array}$$

as the local monodromy near each singular fibers is trivial [Bri66]. Concretely, one takes a small resolution of each of the 32 ordinary singularities of  $\mathcal{X}'$ . This entails replacing each singularity with a  $\mathbb{P}^1$ ; étale locally, small resolutions of  $\{x_0x_1 + x_2^2 = u^2\}$  can be obtained by blowing up either of the planes  $\{x_0 = x_2 - u = 0\}$  or  $\{x_1 = x_2 - u = 0\}$ . The irreducibility of the monodromy representation means  $\mathcal{Y}$  is not even locally projective over  $B$ . It follows that  $\mathcal{Y}$  exists as an algebraic space but not as a scheme.

## 2 Results for general rationally connected varieties

**Basic properties of rationally connected varieties** There are numerous basic references for rationally connected varieties, e.g., Kollár's book [Kol96, IV.3], Debarre's book [Deb01, ch. 4], and Bonavero's contribution to this volume. And we should mention the original papers [KMM92b] and [Cam92].

One general point on terminology: A *rational curve* on a variety  $Y$  is the image of a non-constant morphism  $f : \mathbb{P}^1 \rightarrow Y$ . In particular, rational curves are always proper, even when  $Y$  is not proper.

**Definition 2.1.** Let  $Y$  be a smooth variety. It is *rationally connected* (resp. *rationally chain connected*) if there exists a proper flat morphism  $Z \rightarrow W$  over a variety  $W$ , whose fibers are irreducible (resp. connected)

curves of genus zero, and a morphism  $\phi : Z \rightarrow Y$  such that the induced morphism

$$\phi^2 : Z \times_W Z \rightarrow Y \times Y$$

is dominant.

Roughly,  $Y$  is rationally connected if there exists a rational curve  $f : \mathbb{P}^1 \rightarrow Y$  through the generic pair of points  $(y_1, y_2) \in Y \times Y$ .

**Definition 2.2.** Let  $Y$  be a smooth variety. A non-constant morphism  $f : \mathbb{P}^1 \rightarrow Y$  is *free* (resp. *very free*) if we have a decomposition

$$f^*T_Y \simeq \bigoplus_{i=1}^{\dim(Y)} \mathcal{O}_{\mathbb{P}^1}(a_i)$$

with each  $a_i \geq 0$  (resp.  $a_i > 0$ ).

Trivially, every rationally connected variety is rationally chain connected and each variety admitting a very free curve admits a free curve. Slightly less trivial is that every variety admitting a very free curve is in fact rationally connected. For smooth varieties  $Y$  over fields of characteristic zero, we can prove converses to these statements. Generic smoothness implies that the general rational curve in a family dominating  $Y \times Y$  is very free. And smooth rationally chain connected varieties are in fact rationally connected.

**Elements of deformation theory** We recall a basic result we shall use frequently.

Let  $Y$  be a smooth variety. Consider a map  $\{f : (C, x_1, \dots, x_r) \rightarrow Y\}$  defined on a nodal proper connected curve with distinct marked smooth points, with  $f$  non-constant and unramified at the nodes and the marked points. In particular,  $f$  does not contract any irreducible components of  $C$ , so  $\{f : C \rightarrow Y\}$  and  $\{f : (C, x_1, \dots, x_r) \rightarrow Y\}$  are both stable maps. (See the Appendix for background on stable maps.)

Let  $N_f$  denote the *normal sheaf* of  $f$ , defined (see [GHS03, p. 61]) as the unique non-vanishing cohomology group of

$$\mathbb{R}\mathcal{H}om_{\mathcal{O}_C}(f^*\Omega_Y^1 \xrightarrow{df^\dagger} \Omega_C^1, \mathcal{O}_C),$$

which fits into an exact sequence

$$0 \rightarrow \mathcal{H}om_{\mathcal{O}_C}(\ker(df^\dagger), \mathcal{O}_C) \rightarrow N_f \rightarrow \mathcal{E}xt_{\mathcal{O}_C}^1(\operatorname{coker}(df^\dagger), \mathcal{O}_C) \rightarrow 0.$$

When  $C$  is smooth,  $\Omega_C^1$  is invertible and we obtain

$$0 \rightarrow T_C \rightarrow f^*T_Y \rightarrow N_f \rightarrow 0.$$

Generally, the tangent space and obstruction space to the moduli stack of stable maps at  $\{f : C \rightarrow Y\}$  are  $\Gamma(C, N_f)$  and  $H^1(C, N_f)$  respectively. Sometimes, we will abuse notation and write  $N_{C/Y}$  for  $N_f$ .

Let  $\mathcal{I}_{x_1, \dots, x_r}$  denote the ideal sheaf for the marked points. Consider the substack of the stable map space consisting of  $\{f' : (C', x'_1, \dots, x'_r) \rightarrow Y\}$  such that  $f'(x'_i) = f(x_i)$  for each  $i$ . Its tangent space at  $\{f : (C, x_1, \dots, x_r) \rightarrow Y\}$  equals  $\Gamma(C, N_f \otimes \mathcal{I}_{x_1, \dots, x_r})$ ; the obstruction space is  $H^1(C, N_f \otimes \mathcal{I}_{x_1, \dots, x_r})$ .

**Lemma 2.3.** [GHS03, §2] Suppose  $N_f \otimes \mathcal{I}_{x_1, \dots, x_r}$  has no higher cohomology and is generated at the nodes by global sections. Then  $f : C \rightarrow Y$  admits a smoothing  $f'$  with  $f'(x_i) = f(x_i)$  for each  $i$ .

Lemma 2.3 is typically used to show that stable maps admit smoothings provided they have enough free or very free curves among their irreducible components. It can be applied to establish:

**Proposition 2.4.** [Kol96, IV.3.9.4] *Let  $Y$  be a smooth rationally connected variety over  $k$ , an algebraically closed field of characteristic zero.*

*There exists a unique maximal nonempty open subset  $Y^\circ \subset Y$  such that, for any finite collection of points  $y_1, \dots, y_m \in Y^\circ$ , there exists a morphism  $f : \mathbb{P}^1 \rightarrow Y^\circ$  with image containing the points. Any rational curve meeting  $Y^\circ$  is contained in  $Y^\circ$ .*

*If  $Y$  is proper  $Y^\circ = Y$ .*

Currently, no example is known where  $Y^\circ \subsetneq Y$ . Actually, we shall require a slight variation on this result:

**Proposition 2.5.** [Deb03, 2.2] *Given any collection of points  $y_1, \dots, y_m \in Y^\circ$  and non-trivial tangent vectors  $v_j \in T_{y_j}Y^\circ$ , there exists a morphism  $f : \mathbb{P}^1 \rightarrow Y^\circ$  and points  $p_1, \dots, p_m \in \mathbb{P}^1$  such that  $f(p_j) = y_j$  and  $\text{image}(df_{p_j}) = \text{span}(v_j)$  for  $j = 1, \dots, m$ .*

In fact, we can produce a rational curve through an arbitrary curvilinear subscheme of  $Y^\circ$ ; see [HT08a, Prop. 13]. (By definition, a zero-dimensional scheme is *curvilinear* if it can be embedded into a smooth curve.) The existence of rational curves with prescribed tangencies can be deduced from Proposition 2.4. This foreshadows our weak approximation argument.

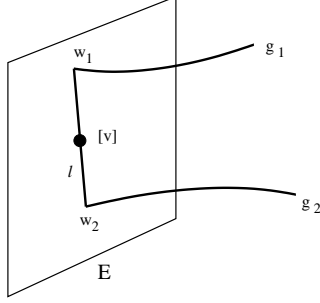


Figure 3: The reducible curve  $C$

*Proof.* Fix  $y \in Y^\circ$  and  $0 \neq v \in T_y Y^\circ$ , and suppose we are given  $f : \mathbb{P}^1 \rightarrow Y^\circ$  with  $f(0) = y$ . Since we may choose  $f$  to pass through a large number of prescribed points in addition to  $y$ , we may assume that  $f$  is very free. Let

$$\tilde{f} : \mathbb{P}^1 \rightarrow \tilde{Y} = \text{Bl}_y(Y)$$

denote the lift to the blow-up of  $Y$  at  $y$ ; let

$$E = \mathbb{P}(T_y Y^\circ) \simeq \mathbb{P}^{\dim(Y)} \subset \tilde{Y}$$

denote the exceptional divisor and  $[v] \in E$  the point corresponding to  $v$ . Since

$$\tilde{f}^* T_{\tilde{Y}} = f^* T_Y \otimes \mathcal{I}_0 \simeq f^* T_Y(-1),$$

the curve  $\tilde{f}$  is at least free. Thus a general deformation  $g_t$  of  $\tilde{f}$  meets  $E$  at a general point.

Consider *two* such deformations  $g_1$  and  $g_2$ , meeting  $E$  at  $w_1$  and  $w_2$  respectively. Choose these in such a way that the line  $\ell \subset E$  joining  $w_1$  and  $w_2$  contains  $[v]$ . Consider the reducible curve

$$C = g_1(\mathbb{P}^1) \cup \ell \cup g_2(\mathbb{P}^1);$$

we analyze the normal bundle  $F := N_{C/\tilde{Y}} \otimes \mathcal{I}_{[v]}$ , corresponding to infinitesimal deformations of  $C \subset \tilde{Y}$  that still contain  $[v]$ . By Lemma 2.3, it suffices to show this is globally generated and has no higher cohomology.

We know that  $F|_{g_i(\mathbb{P}^1)}$  is globally generated with no higher cohomology

because  $g_i$  is free. We compute  $F|_\ell$  using the exact sequences:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \rightarrow & N_{\ell/E} & \rightarrow & N_{\ell/\tilde{Y}} & \rightarrow & N_{E/\tilde{Y}}|_\ell \rightarrow 0 \\
& & \parallel & & \downarrow & & \downarrow \\
0 & \rightarrow & N_{\ell/E} & \rightarrow & N_{C/\tilde{Y}}|_\ell & \rightarrow & Q \rightarrow 0 \\
& & & & \downarrow & & \downarrow \\
& & & & \mathcal{O}_{\{w_1, w_2\}} & = & \mathcal{O}_{\{w_1, w_2\}} \\
& & & & \downarrow & & \downarrow \\
& & & & 0 & & 0
\end{array}$$

Since  $\ell$  is a line and  $E$  is exceptional, we know

$$N_{\ell/E} \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus \dim(Y)-2}, \quad N_{E/\tilde{Y}}|_\ell \simeq \mathcal{O}_{\mathbb{P}^1}(-1).$$

However, this negativity is overcome by the contribution of the nodes  $w_1$  and  $w_2$ , which implies that  $Q \simeq \mathcal{O}_{\mathbb{P}^1}(1)$ . Thus  $N_{C/\tilde{Y}}|_\ell \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus \dim(Y)-1}$  and  $F|_\ell \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus \dim(Y)-1}$ .  $\square$

**Kollár-Miyaoka-Mori and Graber-Harris-Starr Theorems** The following theorem is a forerunner of weak approximation results:

**Theorem 2.6.** *[KMM92b, 2.13] Let  $X$  be a smooth proper rationally connected variety over  $F = k(B)$  with model  $\pi : \mathcal{X} \rightarrow B$ . Assume  $X(F) \neq \emptyset$ , i.e., there exists a section  $s : B \rightarrow \mathcal{X}$ . Given distinct places  $b_1, \dots, b_r$  of good reduction for  $\mathcal{X}$  and points  $x_i \in \mathcal{X}_{b_i}$  for  $i = 1, \dots, r$ , there exists a section  $s' : B \rightarrow \mathcal{X}$  with  $s'(b_i) = x_i$  for each  $i$ .*

We shall sketch the main ideas of the proof below. It took another ten years to remove the assumption on the existence of the section:

**Theorem 2.7.** *[GHS03] If  $X$  is a smooth proper rationally connected variety over  $F = k(B)$  then  $X(F) \neq \emptyset$ .*

A good survey of this important result is [Sta09].

## Weak approximation at places of good reduction

**Theorem 2.8.** *[HT06] Let  $X$  be a smooth proper rationally connected variety over  $F = k(B)$ . Then weak approximation holds at places of good reduction for  $X$ .*

We sketch the ideas of the proof in the next two paragraphs. Throughout, let  $d = \dim(X)$  and  $\mathcal{X} \rightarrow B$  a good model for  $X$ ; fix distinct places  $b_1, \dots, b_r \in B$  such that  $\mathcal{X}_{b_1}, \dots, \mathcal{X}_{b_r}$  are smooth. Let  $s : B \rightarrow \mathcal{X}$  denote the section coming from the Graber-Harris-Starr theorem. Our argument proceeds by induction on  $N$ , the order of the jets we seek to approximate.

**The base case** The base ( $N = 0$ ) case is just the Kollár-Miyaoka-Mori theorem (Theorem 2.6), which we sketch for completeness.

We start with a preparation step, which is a key ingredient of the Graber-Harris-Starr theorem. Basically, we need to show that every rationally connected fibration admits a ‘nice section’:

**Proposition 2.9.** *[GHS03, §2] Let  $\mathcal{X} \rightarrow B$  be as above, with section  $s : B \rightarrow \mathcal{X}$ . Then there exists  $s' : B \rightarrow \mathcal{X}$  such that the normal bundle  $N_{s'}$  is globally generated, with vanishing higher cohomology.*

Here is the idea: Let  $U \subset B$  denote the places of good reduction. Given any finite collection  $b'_1, \dots, b'_q \in U$  and non-trivial vertical tangent vectors  $v_i \in T_{s(b'_i)}\mathcal{X}_{b'_i}$ , Proposition 2.5 gives very free curves  $f_i : \mathbb{P}^1 \rightarrow \mathcal{X}_{b'_i}$  passing through  $s(b'_i)$  with tangent  $v_i$ . If we choose sufficiently many  $b'_1, \dots, b'_q$  and appropriate tangent directions  $v_i$ , the union

$$C = s(B) \cup_{s(b'_1)} f_1(\mathbb{P}^1) \cup \dots \cup_{s(b'_q)} f_q(\mathbb{P}^1)$$

has normal bundle that is globally generated and has no higher cohomology; for details, consult the ‘First construction’ and Lemma 2.5 of [GHS03]. Deformation theory (cf. Lemma 2.3) allows us to smooth  $C$  to a section  $s' : B \rightarrow \mathcal{X}$ , whose normal bundle remains globally generated without higher cohomology.

The argument is now fairly straightforward: Fix points  $x_i \in \mathcal{X}_{b_i}$  over places of good reduction  $b_1, \dots, b_r \in B$ . Let  $s' : B \rightarrow \mathcal{X}$  be our nice section and  $g_i : \mathbb{P}^1 \rightarrow \mathcal{X}_{b_i}$  very free curves such that  $g_i(0) = s'(b_i)$  and  $g_i(\infty) = x_i$ . The union

$$C = s'(B) \cup_{s'(b_1)} g_1(\mathbb{P}^1) \cup \dots \cup_{s'(b_r)} g_r(\mathbb{P}^1)$$

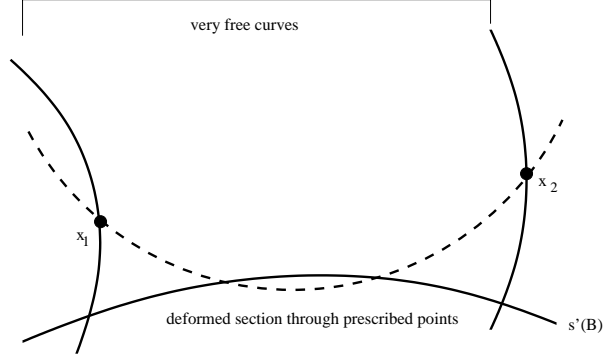


Figure 4: Deformation for the Kollár-Miyaoka-Mori theorem

has normal bundle  $N_{C/\mathcal{X}}$  such that  $N_{C/\mathcal{X}} \otimes \mathcal{I}_{x_1, \dots, x_r}$  is globally generated without higher cohomology. (See [Kol96, II.7.5] for a detailed cohomology analysis.) Lemma 2.3 allows us to smooth  $C$  to a section  $s'' : B \rightarrow \mathcal{X}$  that still contains  $x_1, \dots, x_r$ .

**The inductive step** For simplicity, we describe the procedure in the special case where there is only one place  $b = b_1$ . Let  $\hat{s}^N$  denote the  $N$ -jet of the formal section  $\hat{s}$  we seek to approximate and  $\beta : \mathcal{X}^N \rightarrow \mathcal{X}$  the associated iterated blow-up construction with fiber

$$\mathcal{X}_b^N = \text{Bl}_{\hat{s}(b)}(\mathcal{X}_b) \cup \text{Bl}_{\hat{t}^1(b)}(\mathbb{P}^d) \cup \dots \cup \text{Bl}_{\hat{t}^{N-1}(b)}(\mathbb{P}^d) \cup \mathbb{P}^d.$$

(We retain the notation of Equation (3).) Note that each  $\text{Bl}_{\hat{t}^i(b)}(\mathbb{P}^d)$  is ruled by the proper transforms of the lines through the blown-up point.

Recall Proposition 1.4: We seek sections  $\sigma : B \rightarrow \mathcal{X}^N$  with  $\sigma(b) = x^N$ , where

$$x^N \in \mathbb{P}^d \subset \mathcal{X}_b^N$$

corresponds to the jet  $\hat{s}^N$ . The inductive hypothesis gives a section  $\tau : B \rightarrow \mathcal{X}$  with  $\tau \equiv \hat{s}^N \pmod{\mathfrak{m}_{B,b}^N}$ ; its proper transform  $\tau^N : B \rightarrow \mathcal{X}^N$  satisfies  $y^N := \tau^N(b) \in \mathbb{P}^d$ . We may assume  $\tau^N$  is nice, after another application of Proposition 2.9. And there is nothing to prove unless  $y^N \neq x^N$ .

We construct a chain of rational curves

$$T_0 \cup T_1 \cup \dots \cup T_N$$



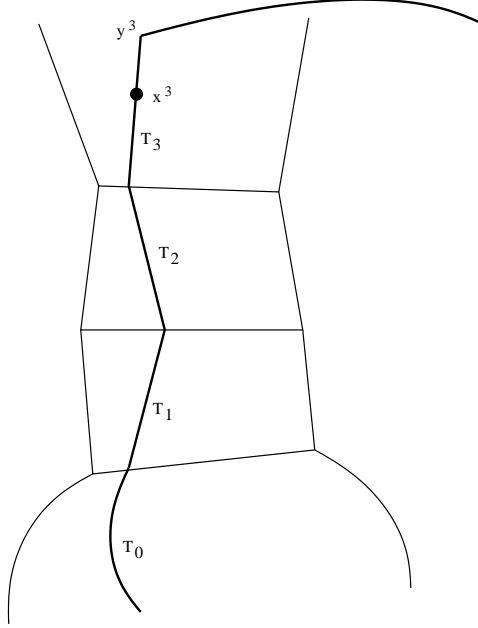


Figure 5: The chain of rational curves for  $N = 3$

with

$$T_0 \subset \text{Bl}_{\hat{s}(b)}(\mathcal{X}_b) \quad T_i \subset \text{Bl}_{\hat{t}^i(b)}(\mathbb{P}^d), 0 < i < N, \quad T_N \subset \mathbb{P}^d,$$

inductively as follows:  $T_N$  is the line joining  $x^N$  and  $y^N$ ,  $T_{N-1}$  is the unique ruling meeting  $T_N$ , etc., and  $T_0$  is a very free curve meeting  $T_1$ . The reducible curve

$$C = \tau^N(B) \cup T_N \cup T_{N-1} \cup \cdots \cup T_1 \cup T_0$$

admits a deformation to a section  $\sigma : B \rightarrow \mathcal{X}^N$  still passing through  $x^N$ .

### Direct generalizations

**Definition 2.10.** A smooth (not necessarily proper) variety  $Y$  is *strongly rationally connected* if through *any* point  $y \in Y$  there passes a very free rational curve.

We rephrase Proposition 2.4 as follows:

**Proposition 2.11.** *Let  $Y'$  be a smooth rationally connected variety over an algebraically closed field of characteristic zero. Then there exists an open,*

nonempty subset  $Y^\circ \subset Y'$  that can be characterized as the maximal strongly rationally connected subset of  $Y'$ .

*Proof.* Let  $Y^\circ$  denote the set produced from Proposition 2.4; recall that any rational curve meeting  $Y^\circ$  is contained in  $Y^\circ$ .

We have already seen (in the course of the proof of Proposition 2.5) that  $Y^\circ$  is strongly rationally connected: The existence of curves through arbitrary finite collections of points means that there are very free curves through any  $y \in Y^\circ$ .

We show that  $Y^\circ$  is maximal among strongly rationally connected subsets of  $Y'$ . If there exists such a subset containing  $y \in Y'$  then there exists a very free curve through  $y$ , i.e., one joining  $y$  to a general point of  $Y'$ . In particular, such a curve necessarily meets  $Y^\circ$  and thus is contained in  $Y^\circ$ .  $\square$

The proof of Theorem 2.8 generalizes directly as follows:

**Theorem 2.12.** *Suppose  $S \subset B$  is a finite set and  $B^\circ = B \setminus S$ . Let  $\pi : \mathcal{Y} \rightarrow B^\circ$  be a smooth morphism with strongly rationally connected fibers. Suppose that  $\pi$  admits a section. For any collection  $J$  of jet data for  $\mathcal{Y}$  over  $B^\circ$  (cf. (2)), there exists a section  $s : B^\circ \rightarrow \mathcal{Y}$  with those jet data.*

This can be successfully applied to certain kinds of singular fibers, e.g., where  $\mathcal{X} \rightarrow B$  is a regular model such that  $\mathcal{X}^{sm} \rightarrow B$  has strongly rationally connected fibers. We shall offer specific examples in Section 3 below.

**A conjecture** The Kollár-Miyaoka-Mori theorem and the weak approximation results sketched above motivate the following general assertion:

*Conjecture 2.13* (Weak approximation for rationally connected varieties). [HT06] Let  $X$  be a rationally connected variety over  $k(B)$ , the function field of a curve  $B$  over  $k$ , an algebraically closed field of characteristic zero. Then  $X$  satisfies weak approximation.

The main technical challenge is the singular fibers. In Section 3, we shall survey situations where the fibers can be successfully analyzed, or where the global geometry ensures weak approximation.

**Converse theorems** We have seen that rationally connected varieties often satisfy weak approximation; in many cases, the converse also holds. The results in this section originate from conversations with Jason Starr.

We start with a purely geometric result:

**Theorem 2.14.** *Let  $X$  be a smooth projective variety over  $k(B)$ , with regular model  $\pi : \mathcal{X} \rightarrow B$ . Suppose that  $\mathcal{X}$  admits a section  $s : B \rightarrow \mathcal{X}$  with the following property: Given general points  $b, b' \in B$  and general points  $x \in \mathcal{X}_b$  and  $x' \in \mathcal{X}_{b'}$ , there exists a deformation  $s'$  of  $s$  such that  $s'(b) = x$  and  $s'(b') = x'$ . Then  $X$  is rationally connected.*

The key hypothesis can be expressed nicely in terms of stable maps (see the Appendix): Let  $\beta = [s(B)] \in H_2(\mathcal{X}_{\mathbb{C}}, \mathbb{Z})$  and  $\overline{\mathcal{M}}'_{g(B), 2}(\mathcal{X}, \beta)$  the irreducible component of the stable map space containing  $\{s : (B, b_1, b_2) \rightarrow \mathcal{X}\}$ , where  $b_1, b_2 \in B$  are general marked points. We assume the evaluation mapping

$$\begin{aligned} \text{ev} : \overline{\mathcal{M}}'_{g(b), 2}(\mathcal{X}, \beta) &\rightarrow \mathcal{X} \times \mathcal{X} \\ \{f : (C, c_1, c_2) \rightarrow \mathcal{X}\} &\mapsto (f(c_1), f(c_2)) \end{aligned} \quad (4)$$

is dominant.

*Proof.* Suppose we have a section  $s' : B \rightarrow \mathcal{X}$  taking general values  $x \in \mathcal{X}_b$  and  $x' \in \mathcal{X}_{b'}$  at distinct general points  $b, b' \in B$ . This yields a two-pointed stable mapping

$$\{s' : (B, b, b') \rightarrow \mathcal{X}\}$$

with  $\text{ev}(s') = (x, x')$ .

Specialize  $b' \rightsquigarrow b$  and  $x' \rightsquigarrow x_2 \in \mathcal{X}_b$  for  $x_2 \neq x$ . This induces a specialization of our stable map

$$\{s' : (B, b, b') \rightarrow \mathcal{X}\} \rightsquigarrow \{f'' : (B'', p, p_2) \rightarrow \mathcal{X}\},$$

where  $f''(p) = x$  and  $f''(p_2) = x_2$ . Now there is a unique irreducible component  $B \subset B''$  such that the restriction  $s'' = f''|_B$  is a section. Let  $C$  denote the union of irreducible components mapped into  $\mathcal{X}_b$ . We know:

- $C$  is a tree of rational curves;
- $x, x_2 \in f''(C)$ .

Thus we have a chain of rational curves in  $\mathcal{X}_b$  joining  $x$  and  $x_2$ , i.e.,  $\mathcal{X}_b$  is rationally chain connected.  $\square$

It is not hard to formulate infinitesimal criteria for when our hypothesis holds:

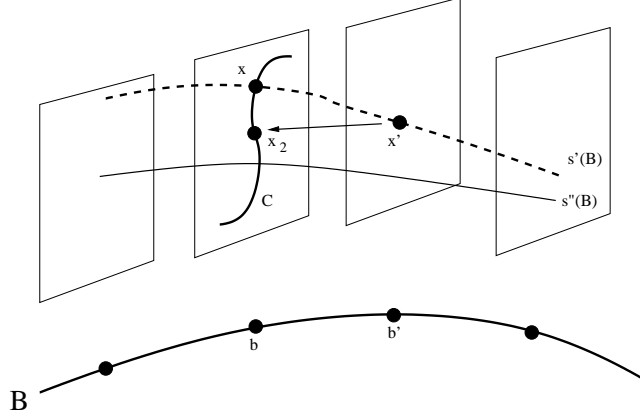


Figure 6: Specializing sections until they break

**Proposition 2.15.** *Let  $X$  be a smooth projective variety over  $k(B)$ , with regular model  $\pi : \mathcal{X} \rightarrow B$  admitting a section  $s : B \rightarrow \mathcal{X}$ . Suppose the normal bundle  $N_s$  has no higher cohomology and, for general  $(b, b') \in B \times B$ , the natural differential*

$$d_{b,b'} : \Gamma(B, N_s) \rightarrow N_s|_b \oplus N_s|_{b'} \simeq T_{s(b)}\mathcal{X}_b \oplus T_{s(b')}\mathcal{X}_{b'}$$

*is surjective. Then (4) is dominant.*

*Proof.* Let  $\overline{\text{Sect}}(\mathcal{X}/B)$  denote the irreducible component of the Hilbert scheme (or algebraic space, if  $\mathcal{X} \rightarrow B$  happens not to be a projective scheme [Art69]) containing  $s(B) \subset \mathcal{X}$ . There is an open subset

$$\text{Sect}(\mathcal{X}/B) \subset \overline{\text{Sect}}(\mathcal{X}/B)$$

containing the *bona fide* sections. This is unobstructed at  $[s(B)]$  by our vanishing assumption, hence smooth of the expected dimension. Given  $b, b' \in B$  distinct, there is an evaluation morphism

$$\begin{aligned} \text{Sect}(\mathcal{X}/B) &\rightarrow \mathcal{X}_b \times \mathcal{X}_{b'} \\ s' &\mapsto (s'(b), s'(b')). \end{aligned}$$

with differential  $d_{b,b'}$  (e.g., [Kol96, I.2]). Since this is surjective, the evaluation morphism is dominant.  $\square$

**Corollary 2.16.** *Assume  $k$  is uncountable. Let  $X$  be a smooth projective variety over  $k(B)$  with regular model  $\pi : \mathcal{X} \rightarrow B$ . Suppose there exist distinct places of good reduction  $b, b' \in B$  such that, for general  $x \in \mathcal{X}_b$  and  $x' \in \mathcal{X}_{b'}$ , there exists a section  $s : B \rightarrow \mathcal{X}$  with  $s(b) = x$  and  $s(b') = x'$ . Then  $X$  is rationally connected.*

See [Kol96, IV.3.6] for a discussion of how the assumption that  $k$  is uncountable allows us to pass from the set-theoretic condition (there exists a section through all points of the fibers) to an algebro-geometric condition (the evaluation morphism is dominant).

### 3 Special cases of weak approximation

We continue to assume  $k$  is algebraically closed of characteristic zero and  $B$  is a smooth projective curve over  $k$ , with function field  $F = k(B)$ .

**Fibration theorems** The following theorem of Colliot-Thélène and Gille [CTG04, 2.2] is fundamental for inductive arguments:

**Theorem 3.1.** *Let  $X^\circ, Y^\circ$  denote smooth varieties over  $F$  and  $f : X^\circ \rightarrow Y^\circ$  a smooth morphism over  $F$  with connected fibers. Assume that*

- *$Y^\circ$  satisfies weak approximation;*
- *for each  $y \in Y^\circ(F)$ ,  $X_y^\circ = f^{-1}(y)$  admits a rational point and satisfies weak approximation.*

*Then  $X^\circ$  satisfies weak approximation.*

Note that the varieties need not be projective; indeed, we shall shrink them so as to satisfy the smoothness hypothesis. The idea of the proof is quite natural: Approximate first in the base, then in the fibers.

Here is a quick but very useful consequence:

**Corollary 3.2.** *Let  $X$  be a smooth projective variety over  $F$ ,  $Y$  a smooth projective variety rational over  $F$ , and  $f : X \rightarrow Y$  a conic fibration, i.e., a morphism whose geometric generic fiber is isomorphic to  $\mathbb{P}^1$ . Then  $X$  satisfies weak approximation.*

Here, take  $Y^\circ \subset Y$  to be the open subset over which  $f$  is smooth and  $X^\circ = f^{-1}(Y^\circ)$ . The base satisfies weak approximation by Corollary 1.7. The fibers are all isomorphic to  $\mathbb{P}^1$ —indeed, conics over  $F$  are automatically split—hence also satisfy weak approximation.

**Classification of surfaces and weak approximation** The birational classification of rational surfaces over a non-closed field is due to Enriques, Manin [Man66], and Iskovskikh [Isk79]. Let  $X$  be a smooth projective surface over  $F$  that is geometrically rational. Assume that  $X$  is *minimal*, in the sense that it admits no birational morphisms  $\phi : X \rightarrow X'$  to a nonsingular projective surface, defined over  $F$ . Then  $X$  is isomorphic to one of the following:

- $\mathbb{P}^2$  or a quadric  $Q \subset \mathbb{P}^3$ ;
- a Del Pezzo surface with  $\text{Pic}(X) = \mathbb{Z}K_X$ , of degree  $d = K_X \cdot K_X$ ;
- a conic bundle over  $\mathbb{P}^1$  with  $\text{Pic}(X) \simeq \mathbb{Z} \oplus \mathbb{Z}$ .

When  $X = \mathbb{P}^2$ , a quadric, or a Del Pezzo surface of degree  $d \geq 5$ , then  $X$  is rational and satisfies weak approximation by Corollary 1.7 (cf. [Has09, §3.5]). The conic bundle cases are covered by Corollary 3.2. The case of degree four Del Pezzo surfaces falls under this category (see [CTG04, §2]): If  $X$  is a degree four Del Pezzo surface over  $F = k(B)$  then  $X(F)$  is Zariski dense by the Kollár-Miyaoka-Mori theorem (Theorem 2.6). For suitable  $x \in X(F)$ , the blow-up  $X' = \text{Bl}_x(X)$  is a cubic surface containing a line  $\ell$ , i.e., the exceptional divisor over  $x$ . Projecting from  $\ell$  gives a conic bundle structure

$$\pi_\ell : X' \rightarrow \mathbb{P}^1;$$

weak approximation for  $X'$  and  $X$  therefore follows.

Thus to complete the case of rational surfaces, it remains to prove weak approximation for Del Pezzo surfaces of degree 3, 2, and 1.

*Remark 3.3* (suggested by Colliot-Thélène). Some of the discussion of degree four del Pezzo surfaces above extends to more general fields. If such a surface admits a rational point not lying on a line then it is unirational [Man86, Thm. 29.4]. And over an infinite perfect field, a degree four del Pezzo surface with a rational point admits a rational point not lying on a line (cf. [Man86, Thm. 30.1]). In particular, we can find a rational point such that projection from that point yields a smooth cubic surface.

**Hypersurfaces of very low degree** Here is another example of how Theorem 3.1 can be profitably applied:

**Theorem 3.4.** [HT09] *Let  $X \subset \mathbb{P}^n$  be a smooth hypersurface of degree  $d$ . Define  $\phi : \mathbb{N} \rightarrow \mathbb{N}$  by the recursive formula*

$$\phi(1) = 1, \quad \phi(d) = \binom{\phi(d-1) + d - 1}{\phi(d-1)}, d > 1.$$

*Then  $X$  satisfies weak approximation if  $n \geq \phi(d)$ .*

Tabulating

$$\begin{array}{c|cccc} d & 1 & 2 & 3 & 4 \\ \hline \phi(d) & 1 & 2 & 6 & 84 \end{array}$$

we see that  $\phi(d) \gg d^2$  for  $d \geq 4$ . Theorems 4.1 and 4.7 below give stronger results for  $d \geq 4$ . We therefore focus on Theorem 3.4 in the special case  $d = 3$ .

Here is the idea: Let  $F_1(X) \subset \mathbb{G}(1, n)$  denote the variety of lines on  $X$ , which is smooth of the expected dimension [AK77, 1.12]. An application of the adjunction formula shows

$$\omega_{F_1(X)} \simeq \mathcal{O}_X(5 - n),$$

so  $F_1(X)$  has ample anticanonical class (‘the Fano variety of lines is Fano’). It follows [KMM92a, Thm. 0.1] that  $F_1(X)$  is rationally connected and thus has an  $F$ -rational point  $\Lambda$  by the Graber-Harris-Starr Theorem (Theorem 2.7).

Consider the projection from  $\Lambda$

$$\pi_\Lambda : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-2}$$

and the induced

$$\pi_\Lambda : \mathrm{Bl}_\Lambda X \rightarrow \mathbb{P}^{n-2},$$

which is a conic bundle. Corollary 3.2 implies  $X$  satisfies weak approximation.

**Cubic surfaces with mild singular fibers** We describe situations where Theorem 2.12 applies to cubic surfaces:

**Proposition 3.5.** [HT08a, §5] *Let  $Y$  be a cubic surface with only rational double points. Then the smooth locus  $Y^{sm} \subset Y$  is strongly rationally connected.*

We sketch this in the special case where  $Y$  has a single ordinary double point  $y_0$ .

*Proof.* We apply Proposition 2.4: Any rational curve in  $Y^{sm}$  that meets the maximal strongly rationally connected subset of  $Y^{sm}$  is contained in that subset. Thus given  $y \in Y^{sm}$ , it suffices to exhibit a rational curve joining  $y$  to a general point  $y' \in Y^{sm}$ . The fact that  $y'$  is general implies

- If  $\ell$  is the line containing  $y$  and  $y'$ , then  $\ell$  meets  $Y$  transversally at three distinct points  $\{y, y', y''\}$ .
- There are no lines on  $Y$  through  $y'$  or  $y''$ .
- There exists no hyperplane section containing  $y'$  or  $y''$ , but not containing  $y_0$ , and consisting of a line and a conic meeting tangentially.

Consider the pencil of hyperplane sections containing  $\ell$ , which induces an elliptic fibration

$$\varphi : \text{Bl}_{y_0, y, y', y''} Y \rightarrow \mathbb{P}^1.$$

It suffices to exhibit an irreducible singular fiber of  $\varphi$ . Now  $\varphi$  has a total of twelve singular fibers, counted with multiplicities.

First, consider the fiber  $F_0$  corresponding to the hyperplane section  $H_0$  through  $y_0$ . The possibilities for  $H_0$  are:

1. an irreducible plane cubic with a node at  $y_0$ ;
2. an irreducible plane cubic with a cusp at  $y_0$ ;
3. the union of a line and a conic meeting transversally at  $y_0$ ;
4. the union of a line and a conic meeting tangentially at  $y_0$ .

Note that  $F_0$  consists of the exceptional curve over  $y_0$  and the proper transform of  $H_0$ . The corresponding possibilities are:

1. the union of two  $\mathbb{P}^1$ 's meeting transversally in two points;
2. the union of two  $\mathbb{P}^1$ 's meeting tangentially;
3. the union of three  $\mathbb{P}^1$ 's meeting pairwise transversally;
4. the union of three  $\mathbb{P}^1$ 's coincident at a point.



The multiplicity of  $F_0$  is 2, 3, 3, and 4 respectively; these are computed by summing the Milnor numbers of the corresponding singularities [Tei75, 2.8.3].

There might be up to three additional reducible fibers beyond  $F_0$ ; these are unions of lines through  $y$  and conics through  $y'$  and  $y''$ , meeting transversally. Altogether, these contribute at most six to the multiplicity count. In order to account for all twelve singular fibers, there must be at least one that is irreducible and disjoint from  $y_0$ .  $\square$

Here is the application to weak approximation:

**Proposition 3.6.** *Let  $X$  be a cubic surface over  $F = k(B)$  and  $\pi : \mathcal{X} \rightarrow B$  a regular model. Suppose that  $S \subset B$  is a finite set with complement  $B^\circ$ , chosen such that for  $b \in B^\circ$  of bad reduction,  $\mathcal{X}_b$  is a cubic surface with rational double points. Then  $X$  satisfies weak approximation at places of  $B^\circ$ .*

Here we apply Theorem 2.12 to the open subset  $\mathcal{Y} \subset \mathcal{X} \times_B B^\circ$  where  $\pi$  is smooth. Proposition 3.5 gives strong rational connectedness; Theorem 2.7 gives the required section.

**Further applications to mild singular fibers** We list further cases where this line of reasoning applies. Let  $Y$  be a projective variety with smooth locus  $Y^{sm}$ . Then  $Y^{sm}$  is strongly rationally connected provided

1.  $Y$  is a degree two Del Pezzo surface with certain types of rational double points; this is due to Knecht [Kne07, Kne08].
2.  $Y$  is a log Del Pezzo surface, including surfaces with quotient singularities and ample anticanonical class; this is a result of C. Xu [Xu09a], and is applied to weak approximation questions in [Xu09b].
3.  $Y \subset \mathbb{P}^n$  is a hypersurface of degree  $d \leq n$  with isolated terminal singularities [HT08a, §6]; this includes hypersurfaces of dimension  $\geq 3$  with ordinary double points, i.e., with local equation

$$x_1^2 + x_2^2 + \cdots + x_n^2 + \text{higher order terms} = 0.$$

Combining the last example with the discussion of cubic surfaces, we obtain:

**Corollary 3.7** (Weak approximation for general Fano hypersurfaces). *[HT08a] Let  $X \subset \mathbb{P}^n$  be a hypersurface of degree  $d \leq n$  over  $F = k(B)$ , with square-free discriminant. Then  $X$  satisfies weak approximation.*

*Proof.* The condition that the discriminant is square-free implies  $X$  admits a regular projective model such that the singular fibers each have a single ordinary double point. Indeed, the multiplicity of the discriminant at a point equals the sum of the Milnor numbers of the singularities in the corresponding fiber [Tei75]; an isolated singularity has Milnor number one precisely when it is an ordinary double point.  $\square$

**Non-regular models** Let  $X$  be a smooth projective rationally connected variety over  $F = k(B)$  and  $\pi : \mathcal{X} \rightarrow B$  a model. When  $\mathcal{X}$  is regular, we saw in Section 1 that sections of  $\pi$  necessarily factor through  $\mathcal{X}^{sm} \subset \mathcal{X}$ , the locus where  $\pi$  is smooth. However, when  $\mathcal{X}$  is not regular it may admit sections passing through singularities over places of bad reduction.

There are a number of approaches we can take to deal with this eventuality. The most direct is to resolve singularities  $\varrho : \tilde{\mathcal{X}} \rightarrow \mathcal{X}$  to obtain a regular model  $\tilde{\pi} : \tilde{\mathcal{X}} \rightarrow B$ . Sections  $\sigma : B \rightarrow \mathcal{X}$  through singularities lift to sections  $\tilde{\sigma} : B \rightarrow \tilde{\mathcal{X}}$  meeting exceptional divisors of  $\varrho$  lying over those singularities. When this resolution can be performed explicitly, we can attempt to construct sections meeting prescribed exceptional divisors. However, this approach requires precise control over the rational curves in various homology classes of the fibers of  $\tilde{\pi}$ .

Here is a result that can be proven in this way:

**Theorem 3.8.** [HT09, Thm. 18] *Let  $\mathcal{X} \rightarrow B$  be a model of a smooth projective rationally connected variety  $X$ . Assume that for each place  $b \in B$  of bad reduction, the singular fiber  $\mathcal{X}_b$  has the following properties:*

- $\mathcal{X}_b$  has only ordinary double points;
- $\mathcal{X}_b^{sm}$  is strongly rationally connected;
- if  $\rho : \tilde{\mathcal{X}}_b \rightarrow \mathcal{X}_b$  is the blow up of the double points, then for each component  $D$  of the exceptional locus  $\text{Exc}(\rho)$  there exists a rational curve in  $\tilde{\mathcal{X}}_b$  meeting  $D$  at one point transversally and avoiding  $\text{Exc}(\rho) \setminus D$ .

*Then  $\mathcal{X} \rightarrow B$  satisfies weak approximation.*

Unfortunately, this approach is not robust enough to fully prove weak approximation, even for cubic surfaces.

*Example 3.9.* Consider the Cayley cubic surface

$$Y = \{wxy + xyz + yzw + zwx = 0\} \subset \mathbb{P}^3,$$

which has ordinary double points at  $[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0]$ , and  $[0, 0, 0, 1]$ . Let  $\rho : \tilde{Y} \rightarrow Y$  denote the minimal resolution with

$$\text{Exc}(\rho) = D_1 \sqcup D_2 \sqcup D_3 \sqcup D_4, \quad D_j \simeq \mathbb{P}^1, D_j \cdot D_j = -2.$$

We may interpret  $\tilde{Y}$  as a blow-up of  $\mathbb{P}^2$ : Choose four lines  $L_1, L_2, L_3, L_4 \subset \mathbb{P}^2$  in general position, with intersections  $p_{ij} = L_i \cap L_j$ . Then  $\tilde{Y} \simeq \text{Bl}_{p_{12}, \dots, p_{34}}(\mathbb{P}^2)$  with exceptional curves  $E_{12}, \dots, E_{34}$ . Furthermore, the proper transform of  $L_j$  equals  $D_j$ . If  $L$  is the pullback of the hyperplane class of  $\mathbb{P}^2$  to  $\tilde{Y}$  then  $D_j = L - E_{ja} - E_{jb} - E_{jc}$  where  $\{j, a, b, c\} = \{1, 2, 3, 4\}$ . Thus we have

$$[\text{Exc}(\rho)] = [D_1 + D_2 + D_3 + D_4] = 4L - 2(E_{12} + \dots + E_{34}),$$

which is divisible by two. This is incompatible with the last assumption of Theorem 3.8.

## 4 Weak approximation and rationally simply connected varieties

There are large classes of varieties where weak approximation can be proven at all places. Here we focus on the ‘rationally simply connected varieties’ introduced by Barry Mazur and studied systematically by de Jong and Starr [dJS06].

### An easier argument under strong assumptions

**Theorem 4.1.** *Let  $X$  be a smooth projective variety over  $F = k(B)$ . Assume that for each  $m \geq 1$ , there exists a class  $\beta \in H_2(X_{\mathbb{C}}, \mathbb{Z})$  and an irreducible component of the space of Kontsevich stable maps*

$$M \subset \overline{\mathcal{M}}_{0,m}(X, \beta),$$

*with the following properties:*

- $M$  is defined and absolutely irreducible over  $F$ ;

- a general point of  $M$  parametrizes a smooth immersed curve;
- the evaluation morphism

$$\begin{aligned} \text{ev} : M &\rightarrow X^m \\ \{f : (C, p_1, \dots, p_m) \rightarrow X\} &\mapsto (f(p_1), \dots, f(p_m)) \end{aligned}$$

is dominant with rationally connected generic fiber.

Then  $X$  satisfies weak approximation.

The key hypothesis is known to hold for

1.  $X \subset \mathbb{P}^n$  a complete intersection of degrees  $(d_1, \dots, d_r)$  with  $d_1 \geq d_2 \geq \dots \geq d_r \geq 2$ , provided  $n + 1 \geq \sum_{i=1}^r (2d_i^2 - d_i)$  and  $\deg(\beta) \geq 4m - 6$  [dJS06, 1.2].
2.  $X \subset \mathbb{P}^n$  is a general hypersurface of degree  $d$ , provided  $n \geq d^2$  and  $\deg(\beta) \gg 0$  [dJS06, 1.3][Sta06].

These classes of varieties are said to be *strongly rationally simply connected*.

**Corollary 4.2.** *Let  $X \subset \mathbb{P}^n$  be a smooth complete intersection of type  $(d_1, \dots, d_r)$  over  $F$ . If  $n + 1 \geq \sum_{i=1}^r (2d_i^2 - d_i)$  then  $X$  satisfies weak approximation.*

*Proof.* We apply the iterated blow-up construction, as formulated in Observation 1.5. Suppose  $\mathcal{X} \rightarrow B$  is a regular projective model of  $X$ ,  $b_1, \dots, b_r \in B$  distinct points, and  $x_i \in \mathcal{X}_{b_i}^{sm}$ ,  $i = 1, \dots, r$ . We produce a section  $\sigma : B \rightarrow \mathcal{X}$  with  $\sigma(b_i) = x_i$  for  $i = 1, \dots, r$ .

There exists a multisection

$$\begin{array}{ccc} \mathcal{Z} & \subset & \mathcal{X} \\ & \searrow & \downarrow \\ & & B \end{array}$$

such that  $x_1, \dots, x_r \in \mathcal{Z}$  and  $\mathcal{Z} \rightarrow B$  is unramified at those points. Indeed, embed  $\mathcal{X} \subset \mathbb{P}^N$  and take a general one-dimensional complete intersection containing  $x_1, \dots, x_r$ . Set  $m = \deg(\mathcal{Z}/B)$  and let  $Z \subset X$  denote the corresponding zero-cycle of length  $m$ .

Suppose, for the moment, that  $Y := \text{ev}^{-1}(Z)$  is rationally connected with generic point corresponding to a stable map with the following properties:

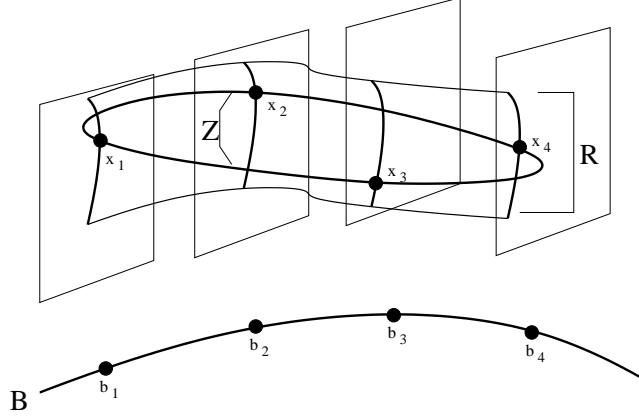


Figure 7: Multisection/ruled surface construction

- the domain is  $\mathbb{P}^1$ ;
- the mapping is an immersion;
- at the marked points the mapping is an embedding, i.e., at these points it is an isomorphism onto its image.

Let  $\mathcal{Y} \rightarrow B$  be a model of  $Y$ . The Graber-Harris-Starr and Kollár-Miyaoka-Mori theorems give a Zariski-dense collection of sections  $\rho : B \rightarrow \mathcal{Y}$ . Given any subvariety  $\Delta \subsetneq Y$ , we may assume  $\rho(B)$  avoids  $\Delta$  in the generic fiber. In particular,  $\rho$  corresponds to an immersed rational curve

$$Z \subset R \subset X$$

with the properties prescribed above.

Let  $\mathcal{R} \subset \mathcal{X}$  denote the closure of  $R$ , a ruled surface containing  $\mathcal{Z}$  and thus  $x_1, \dots, x_r$ . Let  $\psi : \tilde{\mathcal{X}} \rightarrow \mathcal{X}$  be an embedded resolution of singularities for  $\mathcal{R}$ , with proper transform  $\tilde{\mathcal{R}} \subset \tilde{\mathcal{X}}$ . Since  $R$  is smooth along  $Z$ ,  $\psi$  is an isomorphism over  $Z$ ; let  $\tilde{\mathcal{Z}} \subset \tilde{\mathcal{R}}$  denote the closure of  $Z$ . Now  $\tilde{\mathcal{Z}} \rightarrow \mathcal{Z}$  is an isomorphism over the points  $\{x_1, \dots, x_r\}$ ; let  $x'_1, \dots, x'_r \in \tilde{\mathcal{Z}}$  denote their pre-images. Furthermore,  $\tilde{\mathcal{R}} \rightarrow B$  is smooth at these points, because there is a formal section through each  $x'_j$ , e.g.,  $\tilde{\mathcal{Z}}$ . (Recall that  $\tilde{\mathcal{Z}} \rightarrow B$  is unramified at  $x'_j$ .) Weak approximation holds for rational curves, thus there is a section  $\tilde{\sigma} : B \rightarrow \tilde{\mathcal{R}}$  with  $\tilde{\sigma}(b_i) = x'_i$ ;  $\sigma = \psi \circ \tilde{\sigma}$  is our desired section.

We now illustrate how to ensure that  $Y = \text{ev}^{-1}(Z)$  is rationally connected with generic point corresponding to a smooth rational with an immersion that is an embedding at the marked points. There exists a closed subset  $\Delta' \subsetneq X^m$  where one or more of these conditions fails. We need to choose  $Z$  so as to avoid this subset. However, after attaching a suitably large number of fibral very free curves to  $Z$  (cf. Proposition 2.9 and [GHS03, §2]), we can find a degree  $m$  cycle  $Z' \subset X$  avoiding  $\Delta'$  but still containing  $x_1, \dots, x_m$ . Indeed, let  $b \in B$  denote a general point with  $z_1, \dots, z_m \in \mathcal{Z}_b$ . Then there points  $z'_1, \dots, z'_q \in \mathcal{Z}$  and free curves  $f_i : \mathbb{P}^1 \rightarrow \mathcal{X}_{\pi(z'_i)}$  such that the comb

$$C = Z \cup_{z'_1} f_1(\mathbb{P}^1) \cup \dots \cup_{z'_q} f_q(\mathbb{P}^1)$$

has  $N_{C/X} \otimes \mathcal{I}_{z_1, \dots, z_m, x_1, \dots, x_r}$  globally generated with no higher cohomology. The general deformations of  $C$  through  $x_1, \dots, x_r$  is a multisection meeting  $X$  in a general cycle of length  $m$ .  $\square$

**A harder argument under weaker assumptions** We are grateful to Jason Starr for suggesting this argument.

Before we state our result, we review some facts on mappings from rational curves to smooth varieties, following [dJS06, §3].

We shall use the following basic fact repeatedly: Let  $W$  be an integral scheme of finite type over a field  $F$  of characteristic zero. Suppose there exists a geometrically connected nonempty subscheme  $V \subset W$  defined over  $F$ , along which  $W$  is normal. Then  $W$  is geometrically integral. Indeed, all the geometric irreducible components of  $W$  meet along  $V$ ;  $W$  would fail to be normal along  $V$  if there were more than one.

Let  $Y$  be a smooth proper variety over an algebraically closed field of characteristic zero.

**Proposition 4.3.** *Fix a free curve in  $Y$  and let*

$$\overline{\mathcal{M}}'_{0,0}(Y, \beta) \subset \overline{\mathcal{M}}_{0,0}(Y, \beta)$$

*denote the unique irreducible component of the stable map space containing this curve. Then for each  $m \geq 1$  there is a distinguished irreducible component*

$$\overline{\mathcal{M}}'_{0,0}(Y, m\beta) \subset \overline{\mathcal{M}}_{0,0}(Y, m\beta)$$

*characterized by the following property: For each finite morphism  $g : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  of degree  $m > 0$  and free  $\{f : \mathbb{P}^1 \rightarrow Y\} \in \overline{\mathcal{M}}'_{0,0}(Y, \beta)$ , the composition*

$$\{(f \circ g) : \mathbb{P}^1 \rightarrow Y\} \in \overline{\mathcal{M}}'_{0,0}(Y, m\beta).$$

We write  $\overline{\mathcal{M}}'_{0,n}(Y, m\beta)$  for the corresponding irreducible component of the pointed moduli space.

*Proof.* First, the uniqueness of  $\overline{\mathcal{M}}'_{0,0}(Y, \beta)$  is a consequence of the fact that the stable map space is smooth at a free rational curve, because the normal bundle is a quotient of the pull-back of  $T_Y$ . Fix  $m \geq 1$ . The space of all degree  $m$  morphisms  $\mathbb{P}^1 \rightarrow \mathbb{P}^1$  is irreducible, so there is an irreducible scheme  $R_m$  parametrizing the composed maps  $f \circ g$  described above. Each  $f \circ g : \mathbb{P}^1 \rightarrow Y$  remains free; indeed, if  $f^*T_Y \simeq \bigoplus_{j=1}^r \mathcal{O}_{\mathbb{P}^1}(a_j)$  then  $(f \circ g)^*T_Y \simeq \bigoplus_{j=1}^r \mathcal{O}_{\mathbb{P}^1}(ma_j)$ . Hence  $\overline{\mathcal{M}}_{0,0}(Y, m\beta)$  is smooth along  $R_m$ , and thus admits a distinguished irreducible component containing  $R_m$ .  $\square$

*Remark 4.4.* Suppose  $f : \mathbb{P}^1 \rightarrow Y$  is as described above, but defined over an arbitrary field  $F$  of characteristic zero. Then  $\overline{\mathcal{M}}'_{0,0}(Y, m\beta)$  is defined over  $F$  as well.

**Definition 4.5.** Let  $C$  be a nodal connected curve of genus zero and  $Y$  a smooth proper variety. A morphism  $h : C \rightarrow Y$  is *admissible* if, for each irreducible  $C_i \subset C$ , the restriction  $h|_{C_i}$  is either constant or a free immersion.

Each non-constant admissible morphism admits a deformation to a free curve [Kol96, II.7.6]; the moduli stack of stable maps is smooth at such points (see the argument of [FP97, 5.2]).

**Proposition 4.6.** Let  $\overline{\mathcal{M}}'_{0,0}(Y, \beta)$  denote an irreducible component of the stable map space containing a free curve, such that the evaluation map

$$\text{ev} : \overline{\mathcal{M}}'_{0,1}(Y, \beta) \rightarrow Y$$

is dominant with irreducible geometric generic fiber. Let  $h : C \rightarrow Y$  be an admissible stable map such that, for each irreducible component  $C_i \subset C$  with  $h|_{C_i}$  non-constant, the restriction  $h_i := h|_{C_i} : \mathbb{P}^1 \rightarrow Y$  is in  $\overline{\mathcal{M}}'_{0,0}(Y, \beta)$ . Then

- $h \in \overline{\mathcal{M}}'_{0,0}(Y, m\beta)$ , where  $m$  is the number of non-contracted components;
- the evaluation map

$$\text{ev}_m : \overline{\mathcal{M}}'_{0,1}(Y, m\beta) \rightarrow Y$$

also is dominant with irreducible geometric generic fiber.

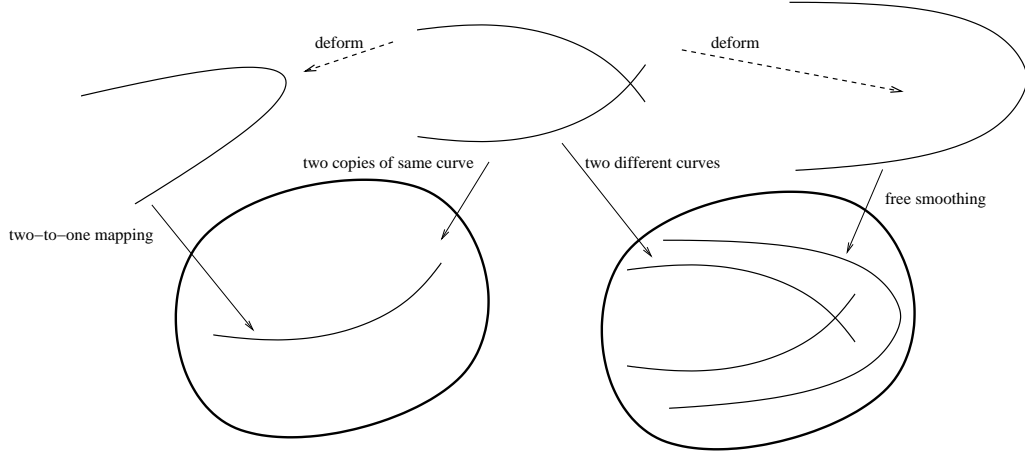


Figure 8: The key degenerations

Thus  $\overline{\mathcal{M}}'_{0,0}(Y, m\beta)$  contains smoothings of chains and trees of  $m$  free curves from  $\overline{\mathcal{M}}'_{0,0}(Y, \beta)$

Proposition 4.6 a special case of [dJS06, 3.5], so we only sketch the proof.

*Proof.* Our proof is by induction on  $m$ . There is nothing to prove in the base case  $m = 1$ , so we focus on the inductive step.

Fix a dense open subset  $U \subset Y$  over which the fibers of

$$\text{ev}_n|_{M'_{0,1}}(Y, n\beta) \rightarrow Y, \quad n = 1, \dots, m-1,$$

are all irreducible of the expected dimension and contain free curves. It follows that for any  $y \in U$  and smooth points  $\{g_1 : (D_1, p_1) \rightarrow Y\} \in \text{ev}_n^{-1}(y)$  and  $\{g_2 : (D_2, p_2) \rightarrow Y\} \in \text{ev}_{m-n}^{-1}(y)$ , the stable map

$$\{g = (g_1, g_2) : D_1 \cup_{p_1=p_2} D_2 \rightarrow Y\} \in \overline{\mathcal{M}}_{0,0}(Y, m\beta)$$

is in the same irreducible component of the stable map space.

This distinguished component must coincide with  $\overline{\mathcal{M}}'_{0,0}(Y, m\beta)$ . Indeed, it suffices to show that the  $g$  constructed in the previous paragraph can be chosen to lie on  $\overline{\mathcal{M}}'_{0,0}(Y, m\beta)$ . We may take  $g_1$  and  $g_2$  to be suitably general branched coverings of a fixed free curve  $\{f : \mathbb{P}^1 \rightarrow Y\} \in M'_{0,0}(Y, \beta)$  through  $y$ . However, the moduli space of stable maps of degree  $m$  from  $\mathbb{P}^1$  to  $\mathbb{P}^1$  is irreducible, and even smooth as a stack. Thus  $g$  deforms to a degree  $m$  covering composed with  $f$ .



Consider our admissible mapping  $h : C \rightarrow Y$ . We may assume  $h$  maps the nodes of  $C$  to  $U$ . Pick  $D_1$  to be an ‘extremal’ irreducible component of  $C$ , i.e., one corresponding to a 1-valent vertex of the dual graph to  $C$ . The stability condition means this is not contracted. Set  $D_2 = \overline{C \setminus D_1}$  and  $p = D_1 \cap D_2$  the disconnecting node joining them. Taking  $g_1 = h|_{D_1}$  and  $g_2 = h|_{D_2}$ , we see that the  $g$  constructed above coincides with  $h$ , and thus is in the desired component.

It remains to show that the generic fiber  $\text{ev}_m^{-1}(y)$  is irreducible. However, it admits a distinguished stratum

$$\text{ev}^{-1}(y) \times \mathcal{M}_{0,m+1},$$

corresponding to attaching  $m$  copies of a fixed mapping

$$\{r : (\mathbb{P}^1, p) \rightarrow Y\} \in \text{ev}^{-1}(y)$$

to a single contracted component  $(\mathbb{P}^1, q, p_1, \dots, p_m)$ . Here the  $j$ th copy is attached to  $p_j$ . This stratum is geometrically irreducible and  $\text{ev}_m^{-1}(y)$  is smooth at its generic point, thus  $\text{ev}_m^{-1}(y)$  is irreducible.  $\square$

**Theorem 4.7.** *Let  $X$  be a smooth projective variety over  $F = k(B)$ . Assume there exists a class  $\beta \in H_2(X_{\mathbb{C}}, \mathbb{Z})$  and an irreducible component of the space of Kontsevich stable maps*

$$M := \overline{\mathcal{M}}'_{0,2}(X, \beta) \subset \overline{\mathcal{M}}_{0,2}(X, \beta)$$

*with the following properties:*

- $M$  is defined and absolutely irreducible over  $F$ ;
- a general point of  $M$  parametrizes an immersed smooth curve;
- the evaluation morphism

$$\begin{aligned} \text{ev} : M &\rightarrow X^2 \\ \{f : (C, p_1, p_2) \rightarrow X\} &\mapsto (f(p_1), f(p_2)) \end{aligned}$$

*is dominant with rationally connected generic fiber.*

*Assume furthermore that the resulting components  $M_m := \overline{\mathcal{M}}'_{0,0}(X, m\beta)$  are rationally connected for each  $m \geq 1$ . Then  $X$  satisfies weak approximation.*

The main hypothesis holds for  $X \subset \mathbb{P}^n$  a complete intersection of degrees  $(d_1, \dots, d_r)$  with  $d_1 \geq d_2 \geq \dots \geq d_r \geq 2$ , provided  $n+1 \geq \sum_{i=1}^r d_i^2$ ,  $n \geq 4$ , and  $\deg(\beta) \geq 2$  [dJS06, 1.1]. These properties are established in the course of proving these varieties are *rationally simply connected*, i.e., for each  $m \geq 2$  there is a canonically-defined irreducible component

$$\overline{\mathcal{M}}'_{0,2}(X, m) \subset \overline{\mathcal{M}}_{0,2}(X, m)$$

of the space of degree  $m$  two-pointed stable maps of genus zero, such that

$$\text{ev} : \overline{\mathcal{M}}'_{0,2}(X, m) \rightarrow X \times X$$

is dominant with rationally connected generic fiber.

**Corollary 4.8.** *Let  $X \subset \mathbb{P}^n$  be a smooth complete intersections of type  $(d_1, \dots, d_r)$  over  $F$ . If  $n+1 \geq \sum_{i=1}^r d_i^2$  then  $X$  satisfies weak approximation.*

*Proof.* The first steps of the argument are identical to Theorem 4.1; thus we retain the notation of its proof, i.e.,  $\mathcal{Z} \rightarrow B$  is the degree  $m$  multisection containing the points we seek to approximate, but otherwise general. Let  $U \subset \mathcal{X} \times_B \mathcal{X}$  denote the open subset over which  $\text{ev}$  has (geometrically) irreducible rationally connected fiber containing a very free immersed rational curve. We may assume  $\mathcal{Z}$  meets the image of  $U$  under the first projection.

Fix a section  $s : B \rightarrow \mathcal{X}$  meeting the image of  $U$  under the section projection. Consider the basechange

$$\mathcal{X} \times_B \mathcal{Z} \rightarrow \mathcal{Z}$$

which has *two* distinguished sections: the base change  $s_{\mathcal{Z}}$  of  $s$  and the diagonal section

$$s' : \mathcal{Z} \hookrightarrow \mathcal{Z} \times_B \mathcal{Z} \hookrightarrow \mathcal{X} \times_B \mathcal{Z}.$$

Let  $F' = k(\mathcal{Z})$  denote the resulting degree  $m$  extension of  $F = k(B)$  and  $M_{F'} = M \times_{\text{Spec}(F)} \text{Spec}(F')$ . We still have that  $\text{ev}^{-1}(\mathcal{U} \times_B \mathcal{Z})$  is rationally connected.

Let  $\overline{\mathcal{M}}'_{0,2}(\mathcal{X} \times_B \mathcal{Z}, \beta)$  denote the irreducible component of the stable map space with generic fiber  $M_{F'}$ ; it is proper over  $\mathcal{Z}$ . The same holds true for

$$\mathcal{P} := \text{ev}^{-1}(s_{\mathcal{Z}}, s') \subset \overline{\mathcal{M}}'_{0,2}(\mathcal{X} \times_B \mathcal{Z}, \beta),$$

where here we take the relative evaluation map. Furthermore, by our assumption this is a family of rationally connected varieties.

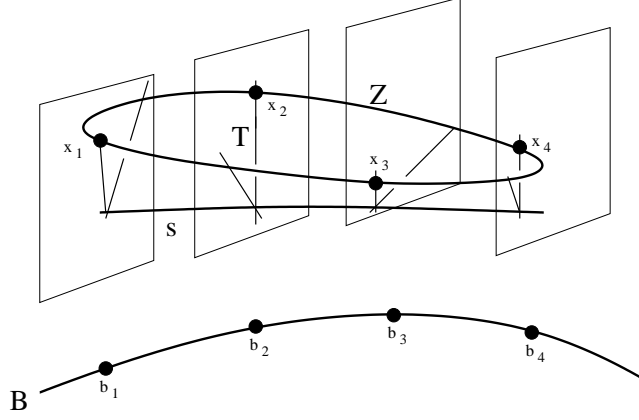


Figure 9: Connecting the multisection to the section

The Graber-Harris-Starr and Kollár-Miyaoka-Mori theorems imply  $\mathcal{P} \rightarrow \mathcal{Z}$  has a section  $\tau : \mathcal{Z} \rightarrow \mathcal{P}$  and these sections are Zariski dense. In particular, we may assume for general  $z \in \mathcal{Z}$ ,  $\tau(z)$  parametrizes a smooth very free rational curve  $T$  joining  $s_{\mathcal{Z}}(z)$  and  $s'(z)$ . We have

$$\begin{array}{ccc} (s_{\mathcal{Z}}(\mathcal{Z}), s'(\mathcal{Z})) & \subset & \mathcal{T}' \subset \mathcal{X} \times_B \mathcal{Z} \\ & \searrow & \downarrow \\ & & \mathcal{Z}, \end{array}$$

where  $\mathcal{T}'$  is the closure of  $T$ . Let  $\mathcal{T}$  denote the image of  $\mathcal{T}'$  under projection to  $\mathcal{X}$ ; this contains both  $s$  and  $\mathcal{Z}$ , which is the projection of  $s'$ .

As in the proof of Theorem 4.1, after blowing up  $\mathcal{X}$  we may assume  $\mathcal{T} \rightarrow B$  is smooth wherever the proper transform  $\mathcal{Z} \rightarrow B$  is étale, and in particular, at  $x_1, \dots, x_r$ .

Let  $b \in B$  be the general point; we realize  $\mathcal{T}_b$  as the image of a stable map  $\{g : C \rightarrow \mathcal{X}_b\} \in \overline{\mathcal{M}}_{0,0}(\mathcal{X}_b, m\beta)$ . First, let

$$C = C_0 \cup C_1 \cup \dots \cup C_m, \quad C_i \simeq \mathbb{P}^1,$$

where  $C_0$  meets  $C_i, i = 1, \dots, m$  in a node  $p_i$  and the  $C_1, \dots, C_m$  are disjoint. Write  $\mathcal{Z}_b = \{z_1, \dots, z_m\}$  and let  $\mathcal{T}_j$  denote the image of  $\mathcal{T}'_{z_j}$  in  $\mathcal{X}_b$ . Since  $\mathcal{T}_j$  is a very free immersed curve, we take  $g|_{C_j} : \mathbb{P}^1 \rightarrow \mathcal{T}_j, j = 1, \dots, m$  to be the normalization, chosen so that  $g(p_j) = s(b)$ . We take  $g|_{C_0}$  to be constant. In the special case where  $m = 2$ , we simply omit the component  $C_0$ ; should

$m = 1$ , we have a section through our prescribed points and there is nothing to prove.

We apply Proposition 4.6, which allows us to interpret  $g$  as a non-stacky, smooth point of  $\overline{\mathcal{M}}'_{0,0}(\mathcal{X}_b, m\beta)$ . Let  $\overline{\mathcal{M}}'_{0,0}(\mathcal{X}, m\beta)$  denote the corresponding irreducible component of the space of stable maps in  $\mathcal{X}$ . This is proper over  $B$  with irreducible fiber  $\overline{\mathcal{M}}'_{0,0}(\mathcal{X}_b, m\beta)$  over  $b$ . Our second hypothesis shows

$$\overline{\mathcal{M}}'_{0,0}(\mathcal{X}, m\beta) \rightarrow B$$

is a rationally connected fibration.

From now on, we tacitly replace  $\overline{\mathcal{M}}'_{0,0}(\mathcal{X}, m\beta)$  with a resolution of singularities that is an isomorphism over the smooth point  $g$ . This resolution is also a rationally connected fibration over  $B$ , and admits a section  $\gamma$  with  $\gamma(b) = g$ . Applying the technique used in Proposition 2.9, we construct a comb  $C$  in  $\overline{\mathcal{M}}'_{0,0}(\mathcal{X}, m\beta)$  with handle  $\gamma(B)$  and teeth in suitable fibers, such that  $N_C \otimes \mathcal{I}_{\gamma(b_1), \dots, \gamma(b_r)}$  is globally generated with vanishing higher cohomology. We deform to get a new section

$$\rho : B \rightarrow \overline{\mathcal{M}}'_{0,0}(\mathcal{X}, m\beta),$$

with  $\rho(b')$  a free immersed curve in  $\overline{\mathcal{M}}'_{0,0}(\mathcal{X}_{b'}, m\beta)$  for general  $b' \in B$ . Applying this argument to the blow-up at  $x_1, \dots, x_r$ , we may even assume  $\rho(b_i) = \gamma(b_i)$  for  $i = 1, \dots, r$ .

Let  $\mathcal{R} \rightarrow B$  denote the ruled surface corresponding to  $\rho$ . By construction, we have  $\mathcal{R}_{b_i} = \mathcal{T}_{b_i}$  is smooth at  $x_i$  for  $i = 1, \dots, r$ . Weak approximation holds in dimension one, e.g., for  $\mathcal{R} \rightarrow B$ , thus we have a section  $\sigma : B \rightarrow \mathcal{R} \hookrightarrow \mathcal{X}$  with  $\sigma(b_i) = x_i$ .  $\square$

**Connections to  $R$ -equivalence** Rational simple connectedness has implications for  $R$ -equivalence as well.

**Definition 4.9.** [Man86] Let  $X$  be a projective variety over a field  $F$ . Two points  $x_1, x_2 \in X(F)$  are *directly  $R$ -equivalent* if there exists a morphism  $f : \mathbb{P}^1 \rightarrow X$  defined over  $F$ , such that  $f(0) = x_1$  and  $f(\infty) = x_2$ . The resulting equivalence relation is known as  *$R$ -equivalence*; the set of equivalence classes is denoted  $X(F)/R$ .

The following nice result of Pirutka [Pir09] is proven using techniques similar to Theorems 4.1 and 4.7:

**Theorem 4.10.** *Let  $F = \mathbb{C}(B)$  denote the function field of a smooth complex curve  $B$ , or the field  $\mathbb{C}((t))$ . Let  $X$  be a smooth complete intersection of  $r$  hypersurfaces in  $\mathbb{P}^n$  of degrees  $d_1, \dots, d_r$ , defined over  $F$ . Assume that  $\sum_{i=1}^r d_i^2 \leq n + 1$ . Then  $X(F)/R = 1$ , i.e., there is a unique  $R$ -equivalence class.*

Here is rough sketch; see [Pir09] for details. Suppose that  $\mathcal{X} \rightarrow B$  is a regular projective model and  $s_1, s_2 : B \rightarrow \mathcal{X}$  are two sections corresponding to  $x_1, x_2 \in X(F)$ . For simplicity, we assume these points are general in  $X$ . The evaluation map

$$\text{ev} : \overline{\mathcal{M}}'_{0,2}(X, m) \rightarrow X \times X$$

is dominant with rationally connected fibers for  $m \geq 2$ . We also have a relative evaluation map

$$\text{ev}_{/B} : \overline{\mathcal{M}}'_{0,2}(\mathcal{X}, m\beta) \rightarrow \mathcal{X} \times_B \mathcal{X}$$

where  $m\beta$  is the class of degree  $m$  fibral curves. Consider the pre-image  $\mathcal{P} = \text{ev}_{/B}^{-1}(s_1, s_2)$ , which is rationally connected (since  $s_1$  and  $s_2$  are general) and thus itself admits a section  $\rho : B \rightarrow \mathcal{P}$ . Indeed, such sections are Zariski-dense in the fibers, and thus correspond to freely immersed rational curves in the generic fiber  $f : \mathbb{P}^1 \rightarrow X$ . This rational curve joins  $x_1$  and  $x_2$ , proving direct  $R$ -equivalence. If  $x_1$  and  $x_2$  are not general, a more sophisticated argument (via specialization) still yields a *chain* of rational curves joining  $x_1$  and  $x_2$ , which suffices to prove their  $R$ -equivalence.

## 5 Questions for further study

1. A. Corti [Cor96] has developed a theory of ‘standard models’ for del Pezzo surfaces of degree  $\geq 2$  over Dedekind schemes; he offers a fairly explicit description of the singularities that arise. Can weak approximation be established using these standard models?
2. While weak and strong approximation coincide for proper varieties, they can differ in general. What can we say about these properties for open log Fano varieties  $Y$  over  $F = k(B)$ , e.g.,  $Y = X \setminus D$  where  $X$  is smooth and proper,  $D \subset X$  is a reduced normal crossings divisor, and  $-(K_X + D)$  is ample?

3. And what about integral points of open log Fano varieties? Suppose  $X$  and  $D$  are as above with compatible models  $\pi : (\mathcal{X}, \mathcal{D}) \rightarrow B$ . Given a finite set  $S \subset B$ , we can consider  $S$ -integral points, defined as sections  $\sigma : B \rightarrow \mathcal{X}$  such that  $\sigma^{-1}(\mathcal{D}) \subset S$  set-theoretically. What approximation properties do these satisfy? (See [HT08b] for Zariski density results in this context.)
4. In the proofs of Theorem 4.1 and 4.7, to what extent can the technical assumptions on rational simple connectedness be weakened? Does it suffice to exhibit some curve class  $\beta$  such that

$$\text{ev} : \mathcal{M}_{0,2}(X, \beta) \rightarrow X \times X$$

is dominant with rationally-connected fibers?

## A Appendix: Stable maps

We continue to work over an algebraically closed field  $k$  of characteristic zero.

**What is a stable map?** We review some basic terminology: A *curve*  $C$  is a reduced connected projective scheme of pure dimension one over  $k$ . The *genus* of a curve is its arithmetic genus  $1 - \chi(\mathcal{O}_C)$ . A point  $p \in C$  is a *node* if either of the following equivalent conditions is satisfied

- the tangent cone of  $C$  at  $p$  is isomorphic to

$$\text{Spec}(k[x, y] / \langle xy \rangle);$$

- $C$  has two smooth branches at  $p$ , meeting transversally.

A *marked point* of  $C$  is a smooth point  $s \in C$  and a *prestable curve with  $n$  marked points*  $(C, s_1, \dots, s_n)$  is a nodal curve  $C$  with  $s_1, \dots, s_n \in C$  such that  $s_i \neq s_j$  when  $i \neq j$ .

If  $C$  is a nodal curve of genus zero then each irreducible component of  $C$  is isomorphic to  $\mathbb{P}^1$ .

Let  $C$  be a nodal curve with normalization  $\nu : C^\nu \rightarrow C$  with connected components  $C_1^\nu, \dots, C_m^\nu$  and images  $C_1 = \nu(C_1^\nu), \dots, C_m = \nu(C_m^\nu)$ . The *dual graph* of  $C$  is a graph with vertices  $\{v_i, i = 1, \dots, m\}$  and edges  $\{e_p\}$  indexed by the nodes  $p \in C$ , i.e., given  $p \in C$  a node with  $\nu^{-1}(p) = \{p', p''\}$

where  $p' \in C_i^\nu$  and  $p'' \in C_j^\nu$ , then the edge  $e_p$  joins  $v_i$  to  $v_j$ . We often put additional structure on this graph, e.g., we can label the vertex  $v_i$  with  $g_i = \text{genus}(C_i^\nu)$ . For a curve with marked point we add a tail (i.e., an edge with just one endpoint) at the vertex corresponding to the component of the normalization containing the point, i.e., if  $s_i$  lies on  $C_j^\nu$  then we attach a tail  $t_i$  to the vertex  $v_j$ .

Let  $C$  be a nodal curve with dualizing sheaf  $\omega_C$  and  $C_i^\nu$  a component of its normalization. Then

$$(\nu^* \omega_C)|_{C_i^\nu} = \omega_{C_i^\nu}(D_i)$$

where  $D_i = \sum_{\text{nodes } p \in C_i} \nu^{-1}(p) \cap C_i^\nu$ .

**Definition A.1.** Let  $(C, p_1, \dots, p_n)$  be a nodal curve with  $n$  marked points. It is *stable* if  $\omega_C(p_1 + \dots + p_n)$  is ample. In combinatorial terms, if  $v_i$  is a vertex of the dual graph with  $g_i = 0$  (resp.  $g_i = 1$ ) then at least three (resp. one) edges or tails are incident to  $v_i$ .

Now fix a scheme  $X$ . A morphism  $f : (C, s_1, \dots, s_n) \rightarrow X$  is called a *prestable map*; it is *stable* if  $\omega_C$  is ample relative to  $f$ , i.e., its restriction to each irreducible component of  $C$  contracted by  $f$  has positive degree. Combinatorially, if  $v_i$  corresponds to a contracted component and  $g_i = 0$  (resp.  $g_i = 1$ ) then at least three (resp. one) edges or tails are incident to  $v_i$ .

Two prestable maps with  $n$  marked points

$$f : (C, s_1, \dots, s_n) \rightarrow X, \quad f' : (C', s'_1, \dots, s'_n) \rightarrow X$$

are *isomorphic* if there is an isomorphism  $\iota : C \rightarrow C'$  over  $X$  with  $\iota(s_i) = s'_i$  for each  $i$ .

Note that precomposing a stable map with an automorphism of the curve gives an isomorphic stable map.

Given a morphism of schemes  $\phi : X \rightarrow B$  and a prestable map  $f : (C, s_1, \dots, s_n) \rightarrow X$ , postcomposing by  $\phi$  clearly gives a prestable map

$$\phi_* f := \phi \circ f : (C, s_1, \dots, s_n) \rightarrow B.$$

**Definition A.2.** Let  $S$  be a scheme of finite type over  $k$ . A *family of prestable curves with  $n$  marked points* consists of a flat proper morphism  $\pi : \mathcal{C} \rightarrow S$  and sections  $s_1, \dots, s_n$  of  $\pi$  such that each geometric fiber

$$(\mathcal{C}_t = \pi^{-1}(t), s_1(t), \dots, s_n(t)), \quad t \in S(k)$$

is prestable with  $n$  marked points.

Given a scheme  $\mathcal{X} \rightarrow S$ , a *prestable map* consists of a family of prestable curves with marked points  $(\mathcal{C}, s_1, \dots, s_n) \rightarrow S$  and a morphism:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{X} \\ & \searrow & \swarrow \\ & S & \end{array}$$

It is a *stable map* if, for each  $t \in S(k)$ , the induced

$$f_t : (\mathcal{C}_t, s_1(t), \dots, s_n(t)) \rightarrow \mathcal{X}_t$$

is a stable map.

Usually we have  $\mathcal{X} = X \times S$  for some scheme  $X$  over  $k$ , in which case we call these families of prestable/stable maps to  $X$ .

Again, given a morphism of schemes over  $S$

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\phi} & \mathcal{B} \\ & \searrow & \swarrow \\ & S & \end{array}$$

and a prestable map  $f : (\mathcal{C}, s_1, \dots, s_n) \rightarrow \mathcal{X}$ , the composition

$$\phi_* f = \phi \circ f : (\mathcal{C}, s_1, \dots, s_n) \rightarrow \mathcal{B}$$

is also prestable.

**Statement of results** The first two results are fairly standard applications of the language of algebraic stacks:

**Theorem A.3** (Existence I). *Let  $X$  be a scheme over  $k$ . There exists a (non-separated) Artin stack  $\mathcal{M}_{g,n}^{ps}(X)$ , locally of finite type over  $k$ , representing families of prestable maps of genus  $g$  curves with  $n$  marked points into  $X$ . Given a morphism  $\phi : X \rightarrow B$  over  $k$ , postcomposition induces a morphism (i.e., a 1-morphism)*

$$\phi_* : \mathcal{M}_{g,n}^{ps}(X) \rightarrow \mathcal{M}_{g,n}^{ps}(B).$$



**Theorem A.4** (Existence II). *Let  $X$  be a proper scheme over  $k$ . There exists an open substack*

$$\overline{\mathcal{M}}_{g,n}(X) \subset \mathcal{M}_{g,n}^{ps}(X)$$

*parametrizing stable maps. Each connected component is a proper Deligne-Mumford stack of finite type over  $k$ . If  $X$  is projective then each connected component of the coarse moduli space is projective.*

The third statement uses slightly different techniques:

**Theorem A.5** (Stabilization). *Let  $X$  be a scheme over  $k$ . Consider the open substack*

$$\mathcal{M}_{g,n}^{\circ}(X) \subset \mathcal{M}_{g,n}^{ps}(X)$$

*parametrizing maps that are non-constant or satisfy  $2g - 2 + n > 0$ . There exists a stabilization morphism*

$$\begin{aligned} \sigma : \mathcal{M}_{g,n}^{\circ}(X) &\rightarrow \overline{\mathcal{M}}_{g,n}(X) \\ \{f : (C, s_1, \dots, s_n) \rightarrow X\} &\mapsto \{f' : (C', s'_1, \dots, s'_n) \rightarrow X\} \end{aligned}$$

*characterized as follows: To obtain  $C'$ , successively contract each irreducible component  $D \subset C$  such that  $f(D) = \{\text{point}\}$  and  $\omega_C(s_1 + \dots + s_n)|_D$  fails to be ample. The morphism  $f$  descends to a morphism  $f' : C' \rightarrow X$ .*

**On existence** We focus on Theorem A.3, as the properness/projectivity assertions of Theorem A.4 are covered in several places in the literature.

We sketch a construction for the relevant stack locally in a neighborhood of

$$f_0 : (C_0, s_1(0), \dots, s_n(0)) \rightarrow X.$$

We abuse terminology, using the term ‘Hilbert scheme’ for the algebraic space parametrizing proper subschemes of a scheme, as constructed in [Art69].

First, choose a line bundle  $L_0$  on  $C_0$  that is very ample with no higher cohomology. Fix a basis for  $\Gamma(C_0, L_0)$  and consider the corresponding embedding  $C_0 \hookrightarrow \mathbb{P}^N$ . Let  $\mathcal{Hilb}_1$  denote the connected component of the Hilbert scheme parametrizing nested subschemes

$$\{s_1(0), \dots, s_n(0)\} \subset C_0 \subset \mathbb{P}^N.$$

Let  $\mathcal{Hilb}_2$  denote the component parametrizing pairs of nested subschemes

$$\begin{aligned} \{s_1(0), \dots, s_n(0)\} &\subset C_0 \subset \mathbb{P}^N \\ \Gamma_{f_0} &\subset C_0 \times X \end{aligned}$$

where  $\Gamma_{f_0}$  is the graph of our stable map. Let  $0 \in \mathcal{Hilb}_2$  denote the distinguished point corresponding to our choice of prestable map and projective embedding of  $C$ . Restricting to a suitable open neighborhood  $0 \in U \subset \mathcal{Hilb}_2$ , we obtain a ‘universal’ prestable map over  $U$ . Precisely, restrict to the  $u \in \mathcal{Hilb}_2$  where

- $s_1(u), \dots, s_n(u)$  are distinct;
- $\mathcal{C}_u$  is nodal;
- $s_i(u) \in \mathcal{C}_u$  is a smooth point;
- $\Gamma_u \subset \mathcal{C}_u \times X$  is the graph of a function.

The relevant prestable map is induced by projection onto  $X$ .

**Claim:** Let  $(S, 0)$  denote a pointed scheme and

$$f : (\mathcal{C}, s_1, \dots, s_n) \rightarrow X \times S$$

a family of prestable maps agreeing with our initial stable map at the distinguished point  $0$ . There exists an étale neighborhood  $S' \rightarrow S$  of  $0$  and a morphism

$$\mu : S' \rightarrow U \subset \mathcal{Hilb}_2$$

such that the pull-back of the universal stable map over  $U$  is isomorphic to the pull back of our family of prestable maps

$$f' : (\mathcal{C}', s'_1, \dots, s'_n) \rightarrow X \times S',$$

where  $\mathcal{C}' = \mathcal{C} \times_S S'$ , the  $s'_j$  are the induced sections, and  $f'$  is the morphism obtained by composition  $f$  with the projection  $\mathcal{C}' \rightarrow \mathcal{C}$ .

Here is the idea of the proof: We choose  $S'$  in such a way that  $\pi' : \mathcal{C}' \rightarrow S'$  admits a relatively very ample line bundle  $\mathcal{L}$  restricting to  $L_0$  at  $C_0$ , and  $\pi'_* \mathcal{L}$  admits a trivialization over  $S'$  restricting to our choice of basis of  $\Gamma(C_0, L_0)$ . Using this data, we get a canonical lift of our prestable map to  $U$ .

Each such  $U$  gives a open neighborhood for

$$[f_0 : (C_0, s_1(0), \dots, s_n(0)) \rightarrow X] \in \mathcal{M}_{g,n}^{ps}(X)$$

in the smooth topology. We would like to use these as the basis of a stack presentation of  $\mathcal{M}_{g,n}^{ps}$ . For any two such open sets  $U'$  and  $U''$ , we must

describe the gluing relation between  $U'$  and  $U''$ : Given universal prestable maps

$$\mathcal{C}' \rightarrow U', \quad f' : (\mathcal{C}', s'_1, \dots, s'_n) \rightarrow X \times U',$$

and

$$\mathcal{C}'' \rightarrow U'', \quad f'' : (\mathcal{C}'', s''_1, \dots, s''_n) \rightarrow X \times U'',$$

we glue fibers over  $u' \in U'$  and  $u'' \in U''$  when there is an isomorphism

$$\iota : \mathcal{C}'_{u'} \rightarrow \mathcal{C}''_{u''}$$

such that  $f'' \circ \iota = f'$ . This is a smooth equivalence relation, so the quotient is an Artin stack (see [LMB00, §4]).

Suppose  $\phi : X \rightarrow B$  is a morphism of projective varieties,

$$f_0 : (C_0, s_1(0), \dots, s_n(0)) \rightarrow X$$

a prestable map to  $X$ , and

$$\phi \circ f_0 : (C_0, s_1(0), \dots, s_n(0)) \rightarrow B$$

the induced map to  $B$ . Let  $\mathcal{Hilb}_2^X$  and  $\mathcal{Hilb}_2^B$  denote the Hilbert schemes constructed as above and  $U \subset \mathcal{Hilb}_2^X$  and  $V \subset \mathcal{Hilb}_2^B$  the associated open subsets. Then composition by  $\phi$  induces morphisms

$$\begin{aligned} \phi_* : U &\rightarrow V \\ \Gamma_f &\mapsto \Gamma_{\phi \circ f} \end{aligned}$$

compatible with the equivalence relations. Thus we get a morphism of the corresponding stacks.

**Discussion of stabilization** Suppose we have a morphism  $S \rightarrow \mathcal{M}_{g,n}^\circ(X)$  corresponding to a prestable map, consisting of a family of prestable curves

$$\pi : (\mathcal{C}, s_1, \dots, s_n) \rightarrow S$$

and a morphism  $f : \mathcal{C} \rightarrow X \times S$  over  $S$ . Fix a line bundle  $\mathcal{L}$  on  $\mathcal{C}$  ample relative to  $\pi$  and thus ample relative to  $f$ .

Suppose that  $\omega_\pi(s_1 + \dots + s_n)$  fails to be ample relative to  $f$ , i.e., some fiber  $\mathcal{C}_t$  admits an irreducible component  $D$  such that  $f(D) = \{\text{point}\}$  and  $\omega_\pi(s_1 + \dots + s_n)$  has negative degree along  $D$ . It is evident that there are

a finite number of curve classes with this property. Choose  $c > 0$  to be the smallest positive number such that the  $\mathbb{Q}$ -divisor

$$\omega_\pi(s_1 + \cdots + s_n) \otimes \mathcal{L}^c$$

is nef relative to  $f$ . Write  $c = a/b$  with  $a, b \in \mathbb{N}$  so that

$$M = \omega_\pi^b(b(s_1 + \cdots + s_n)) \otimes \mathcal{L}^a$$

is Cartier. Note that  $M$  is *trivial* along any component  $D \subset \mathcal{C}_t$  over which it has zero degree.

One can show that for some  $N > 0$ ,  $M^N$  is globally generated and has no higher direct images relative to  $f$ . (We give references below.) In particular  $\oplus_{N \geq 0} f_* M^N$  is finitely generated and we get a morphism

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\beta} & \mathcal{C}' := \text{Proj}(\oplus_{N \geq 0} f_* M^N) \\ & \searrow & \swarrow \\ & X \times S & \end{array}$$

contracting all the components over which  $M$  is trivial. In some sense, this is a strong version of the Kawamata basepoint freeness theorem relative to  $f$ .

*Remark A.6.* A more direct argument can be made when  $X$  is projective [BM96, §3]. Choose an ample divisor  $H$  on  $X$  and consider

$$M = \omega_\pi(p_1 + \cdots + p_n) \otimes f^* \mathcal{O}_X(3H).$$

Now  $M$  generally has base points, e.g., along components  $D \subset \mathcal{C}_t$  contracted by  $f$  that are isomorphic to  $\mathbb{P}^1$  and have just one distinguished point (marked point or node). Nevertheless, one can show there is still a morphism

$$\beta : \mathcal{C} \rightarrow \mathcal{C}' := \text{Proj}(\oplus_{N \geq 0} \pi_* M^N).$$

**Key literature** The concept of stable maps goes back to Kontsevich [Kon95, KM94]. A good algebro-geometric introduction can be found in [FP97]; this survey gives a complete construction when the target space is projective. A stack-theoretic discussion of stable maps can be found in [AV02]: [AV02, §5] addresses existence of the moduli space of stable maps into a projective variety; [AV02, §6] discusses properness and projectivity of the coarse moduli space; the generalization to the case of maps into a proper scheme or

algebraic space can be found in [AV02, 8.4]. An analysis of the stabilization functor—with applications to functoriality of stable maps—can be found in [BM96, §3]. Explicit constructions of stabilization morphisms in related contexts can be found in [Knu83] and [Has03, §3.1,4.1]; these references prove the relative global generation and vanishing asserted above.

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